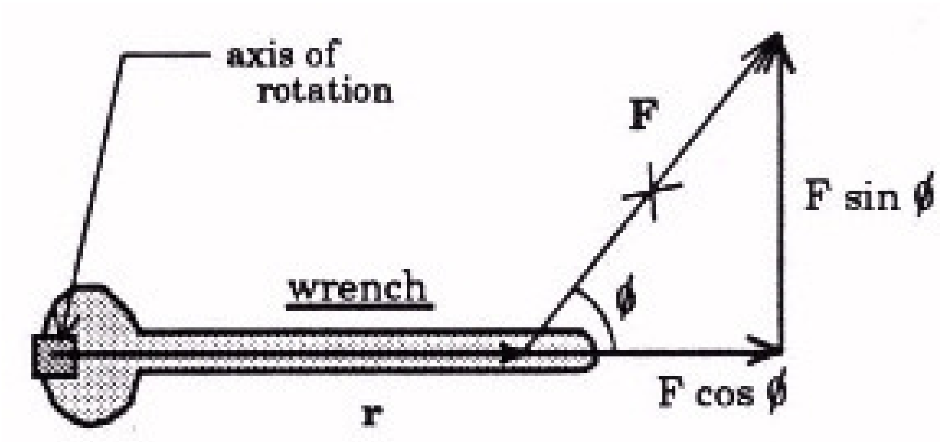
Rotational Motion – Part II

AP Physics C

Torque

So far we have analyzed translational motion in terms of its angular quantities. But we have really only focused on the kinematics and energy. We have yet to add dynamics (Newton's Laws) to the equation..

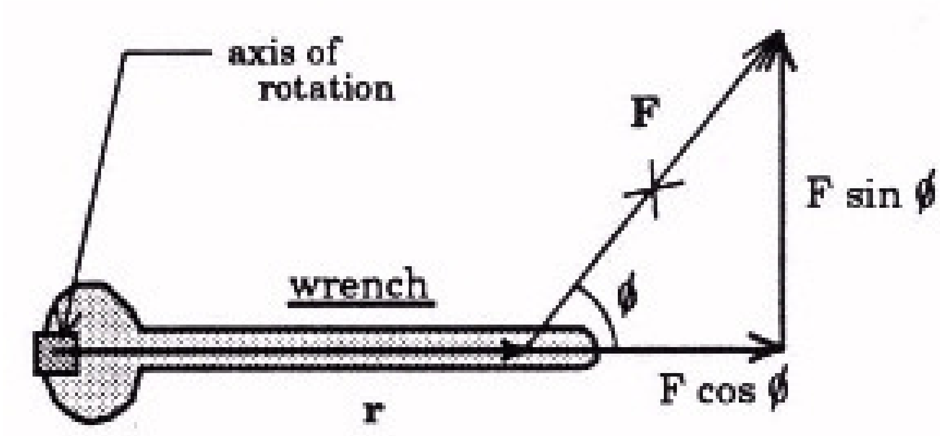


Since Newton's Laws governs how forces act on an object we need to look at how force is applied under angular conditions.

TORQUE is the ANGULAR counterpart to FORCE.

Torque is defined as the Force that is applied TANGENT to the circle rotating around a specific point of rotation.

Torque

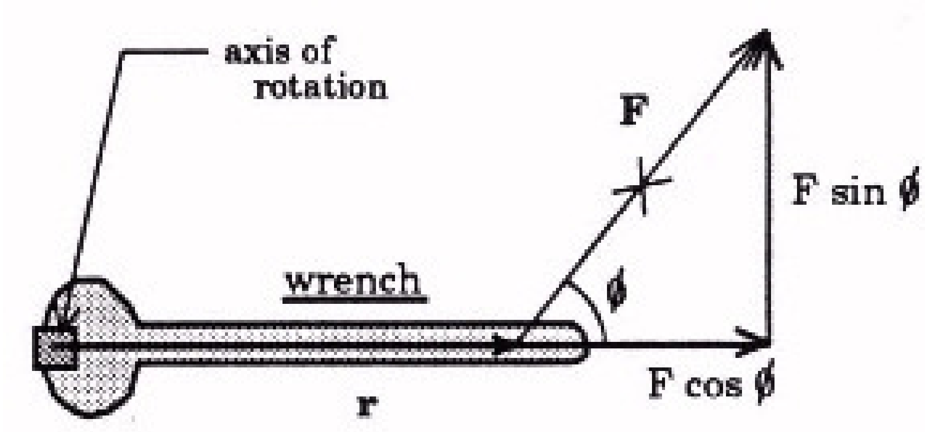


TWO THINGS NEED TO BE UNDERSTOOD:

1) The displacement from a point of rotation is necessary. Can you unscrew a bolt without a wrench? Maybe but it isn't easy. That extra distance AWAY from the point of rotation gives you the extra leverage you need.
THUS we call this distance the **LEVER (EFFORT) ARM (r)** .

2) The Force MUST be ***perpendicular*** to the displacement. Therefore, if the force is at an angle, ***sinθ*** is needed to meet the perpendicular requirement.

Torque is a CROSS PRODUCT



$$\tau = \vec{F} x \vec{r}$$

$$\tau = \vec{F}\vec{r}\sin\theta$$

If the force is truly perpendicular, then the sine of 90 degrees will equal to 1. When the force is applied, the bolt itself moves in or out of the page. In other words, the **FORCE** and **DISPLACEMENT (lever arm)** are in the X/Y plane, but the actual displacement of the BOLT is on the "Z" axis.

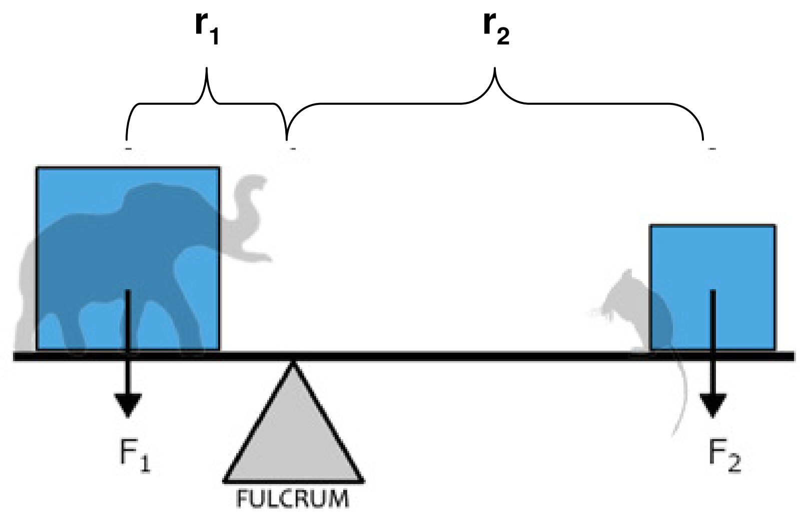
We therefore have what is called, **CROSS PRODUCT.**

- Counterclockwise rotation is considered to be POSITIVE and OUT OF THE PAGE
- Clockwise rotation is considered to be NEGATIVE and INTO THE PAGE.

Static Equilibrium

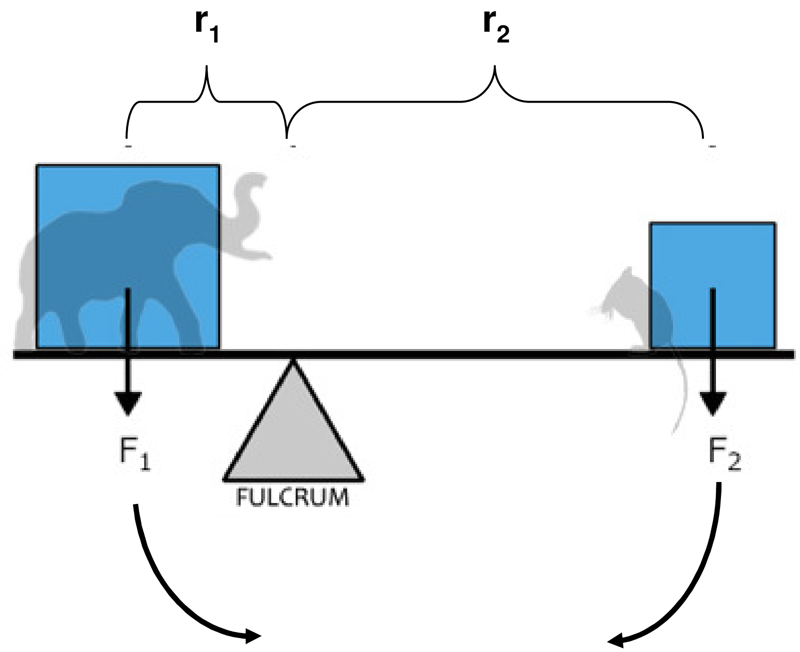
$$\sum \tau_{cw} = \sum \tau_{ccw}$$

According to Newton's first law, if an object is at rest it can be said to be in a state of static equilibrium. In other words, all of the FORCES cancel out to that the net force is equal to zero. ***Since torque is the angular analog to force we can say that if a system is at rest, all of the TORQUES cancel out.***



Static Equilibrium Example

$$\sum \tau_{cw} = \sum \tau_{ccw}$$



Suppose a 4600 kg elephant were placed on a see-saw with a 0.025 kg mouse. The elephant is placed 10 meters from the point of rotation. How far from the point of rotation would the mouse need to be placed so that the system exists in a state of static equilibrium?

$$\tau = Fr\sin\theta,\ \theta = 90,\ \sin 90 = 1$$

$$\sum \tau_{ccw} = \sum \tau_{cw}$$

$$F_{eleph} r_1 = F_{mouse} r_2$$

$$m_{eleph} g r_1 = m_{mouse} g r_2$$

$$m_{eleph} g r_1 = m_{mouse} g r_2$$

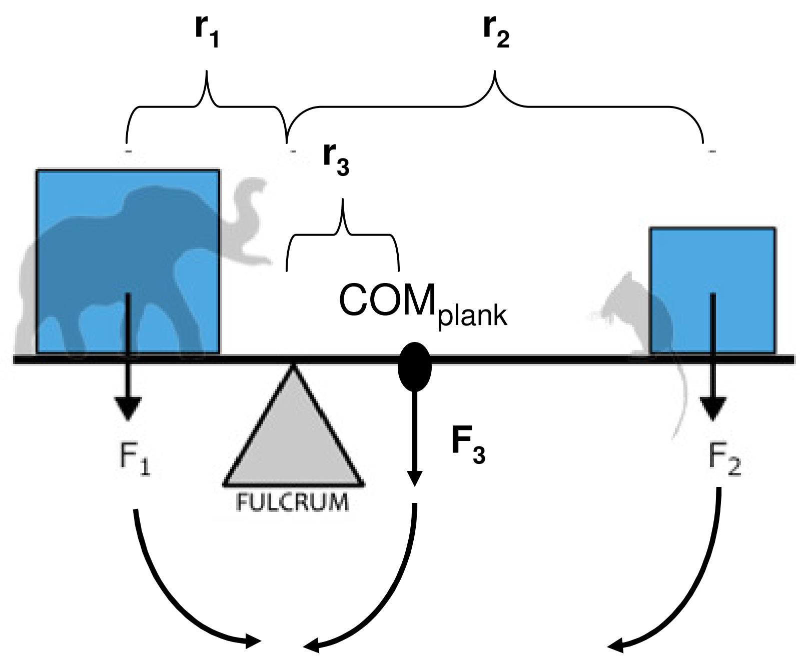
$$(4600)(9.8)(10) = (0.025)(9.8) r_2$$

$r_2 =$ **1.84×10^6 m or 1433 miles (certainly not practical)**

What did we forget to include in the last example?

THE PLANK ITSELF!

If the lever itself has mass, you must include it in the calculations. It's force(or weight in this case) will act at the rods CENTER OF MASS. If the plank was uniform and its COM was in the middle the equation would have looked like this.



$$\tau = Fr\sin\theta,\ \theta = 90,\ \sin 90 = 1$$

$$\sum \tau_{ccw} = \sum \tau_{cw}$$

$$F_{eleph} r_1 = F_{mouse} r_2 + F_{plank} r_3$$

$$m_{eleph} g r_1 = m_{mouse} g r_2 + m_{plank} g r_3$$

Not in static equilibrium?

If an object is NOT at equilibrium, then it must be accelerating. It is then looked at according to Newton's Second Law.

$$F_{net} = ma$$

$$\tau_{net} = I\alpha$$

Under translational conditions a NET FORCE produces an ACCELERATION.

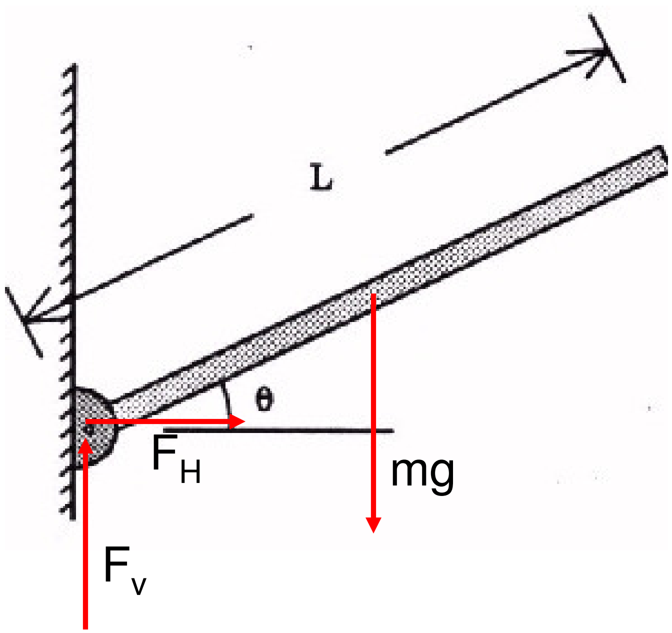
Under Angular Conditions a NET TORQUE produces an ANGULAR ACCELERATION.

This NEW equation for TORQUE is the Rotational Analog to Newton's second Law.

Example

Consider a beam of Length *L*, mass *m*, and moment of inertia (COM) of $1/2mL^2$. It is pinned to a hinge on one end.

Determine the beam's angular acceleration.



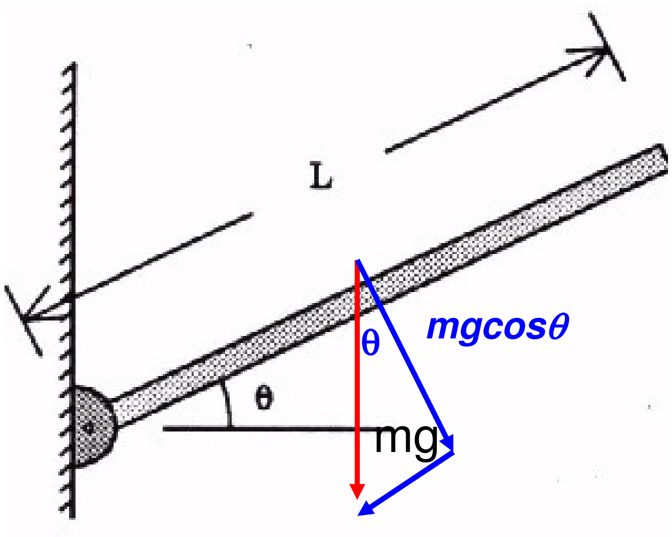
Let's first look at the beam's F.B.D.

There are always vertical and horizontal forces on the pinned end against the hinge holding it to the wall. However, those forces ACT at the point of rotation.

Example

Consider a beam of Length *L*, mass *m*, and moment of inertia (COM) of $1/12mL^2$. It is pinned to a hinge on one end.

Determine the beam's angular acceleration.



$$Fr\sin\theta = \tau = I\alpha$$

$$(mg\cos\theta)(L/2)(1) = I_{pin}\alpha$$

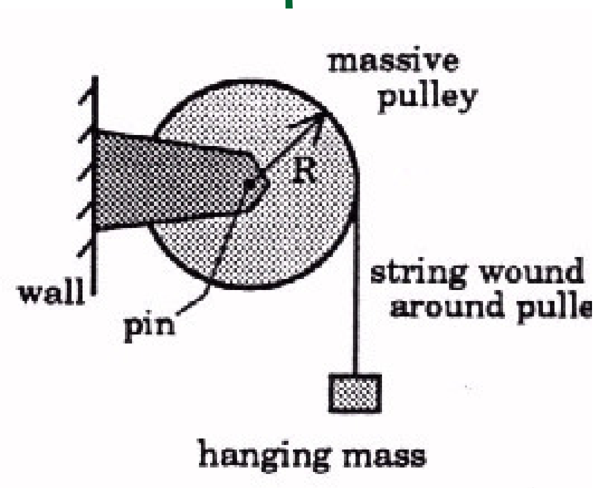
$$(mg\cos\theta)(L/2) = (I_{cm} + md^2)\alpha$$

$$(mg\cos\theta)(L/2) = (1/12\, mL^2 + m(L/2)^2)\alpha$$

$$\alpha = \boxed{\frac{3g\cos\theta}{2L}}$$

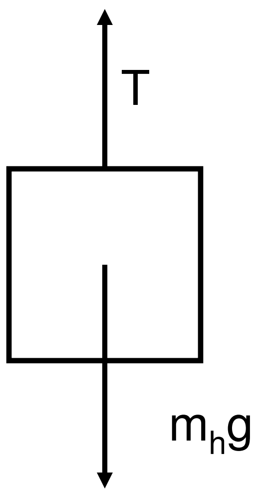
In this case, it was the vertical component of the weight that was perpendicular to the lever arm. Also, we had to use the parallel axis theorem to determine the moment of inertia about the END of the beam.

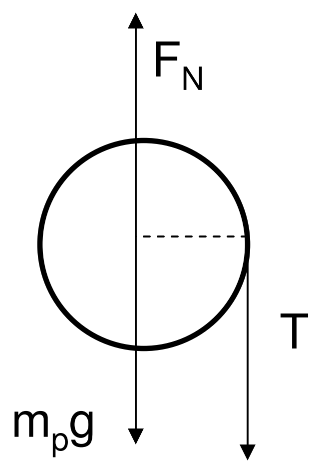
Example



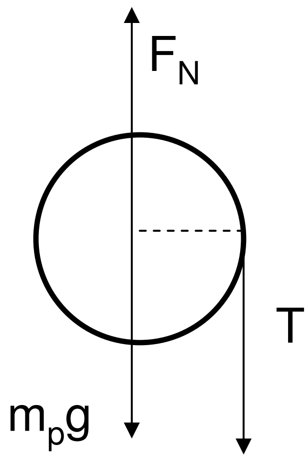
Consider a hanging mass wrapped around a MASSIVE pulley. The hanging mass has weight, ***mg***, the mass of the pulley is $\boldsymbol{m_p}$, the radius is **R**, and the moment of inertia about its center of mass $\mathbf{I_{cm} = 1/2m_pR^2}$. (assuming the pulley is a uniform disk). Determine the acceleration of the hanging mass.

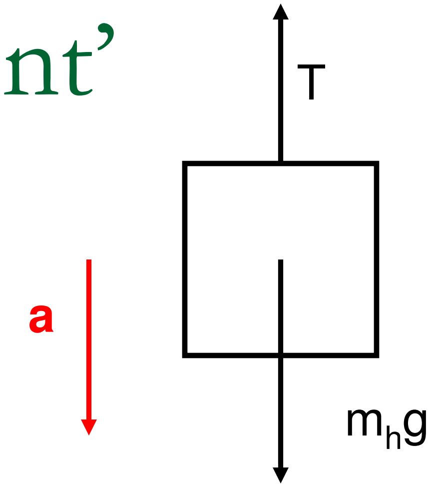
Let's first look at the F.B.D.s for both the pulley and hanging mass





Example cont'



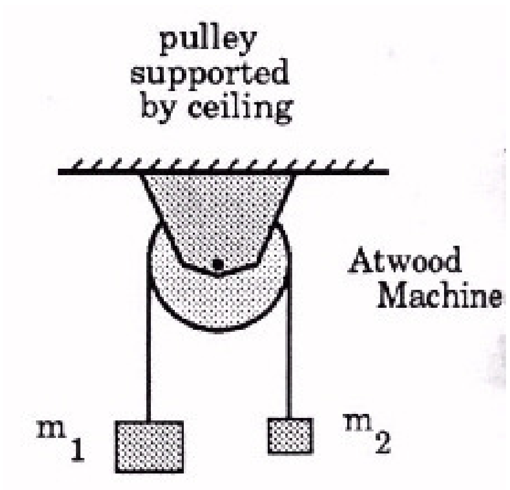


$$F_{Net} = ma$$
$$m_h g - T = m_h a$$
$$T = m_h g - m_h a$$

$$Fr\sin\theta = \tau = I\alpha$$
$$TR = I\alpha, \quad a = \alpha r \quad I_{disk\,@\,CM} = \tfrac{1}{2} m_p R^2$$
$$TR = \tfrac{1}{2} m_p R^2 \left(\frac{a}{R}\right)$$
$$T = \tfrac{1}{2} m_p a$$

$$m_h g - m_h a = T = \tfrac{1}{2} m_p a$$
$$m_h g = \tfrac{1}{2} m_p a + m_h a$$
$$a = \frac{m_h g}{\tfrac{1}{2} m_p + m_h}$$

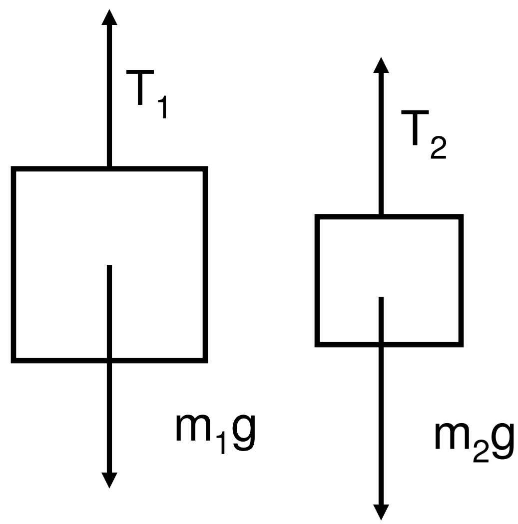
Example

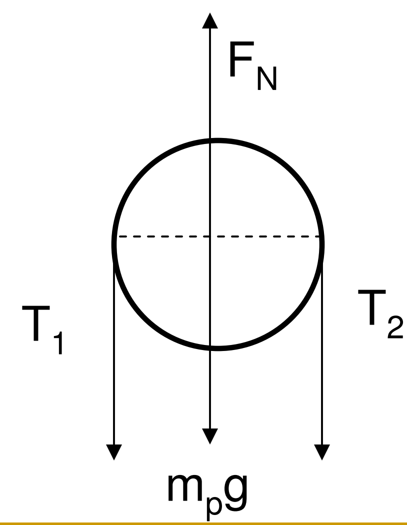


A trickier problem: **Calculate the acceleration of the system:**

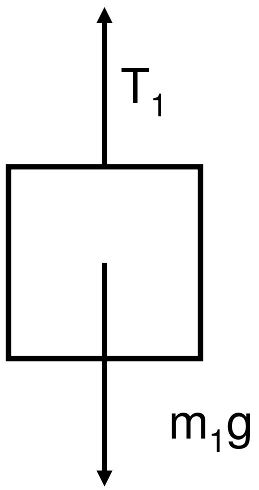
Assume m_1 is more massive than m_2
What you have to understand is that when the PULLEY is massive you <u>cannot</u> assume the tension is the same on both sides.

Let's first look at the F.B.D.s for both the pulley and the hanging masses.

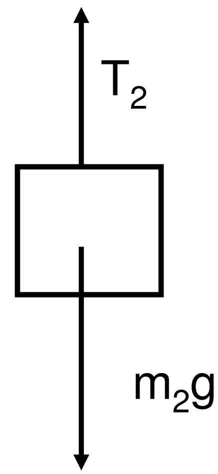




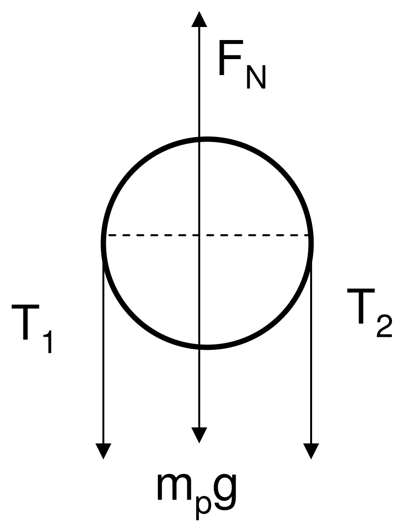
Example cont'



$$F_{Net} = ma$$
$$m_1 g - T_1 = m_1 a$$
$$T_1 = m_1 g - m_1 a$$



$$F_{Net} = ma$$
$$T_2 - m_2 g = m_2 a$$
$$T_2 = m_2 a + m_2 g$$



$$Fr\sin\theta = \sum \tau = I\alpha$$
$$T_1 R - T_2 R = I\alpha, \quad a = \alpha r \quad I_{disk\,@\,CM} = \tfrac{1}{2} m_p R^2$$
$$T_1 R - T_2 R = \tfrac{1}{2} m_p R^2 \left(\frac{a}{R}\right)$$
$$T_1 - T_2 = \tfrac{1}{2} m_p a$$

Example

$$T_1 - T_2 = \tfrac{1}{2} m_p a \quad T_1 = m_1 g - m_1 a \quad T_2 = m_2 a + m_2 g$$

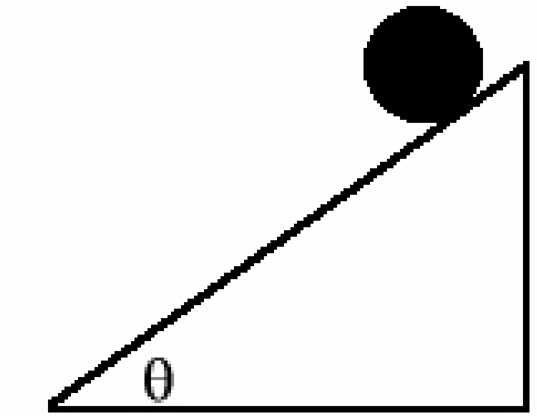
$$m_1 g - m_1 a - (m_2 a + m_2 g) = \tfrac{1}{2} m_p a$$

$$m_1 g - m_1 a - m_2 a - m_2 g = \tfrac{1}{2} m_p a$$

$$m_1 g - m_2 g = m_1 a + m_2 a + \tfrac{1}{2} m_p a$$

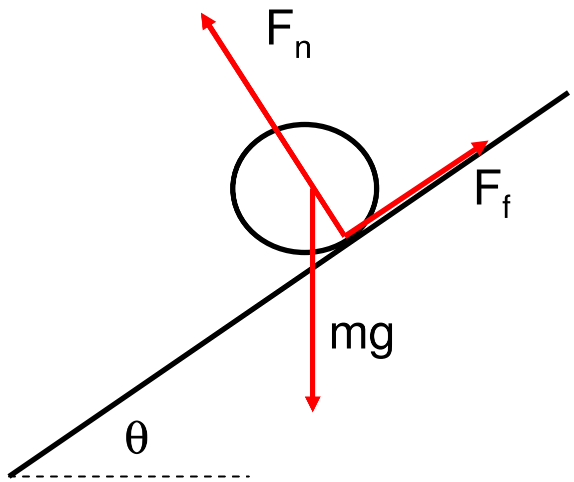
$$a = \frac{m_1 g - m_2 g}{m_1 + m_2 + \tfrac{1}{2} m_p}$$

Example



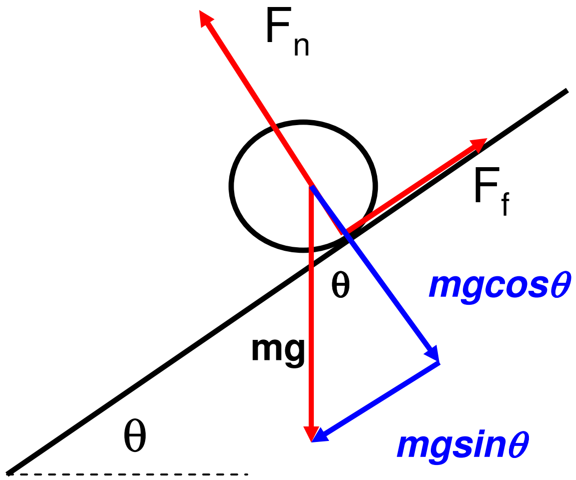
Consider a ball ***rolling*** down a ramp. Calculate the translational acceleration of the ball's center of mass as the ball rolls down. Find the angular acceleration as well. Assume the ball is a solid sphere.

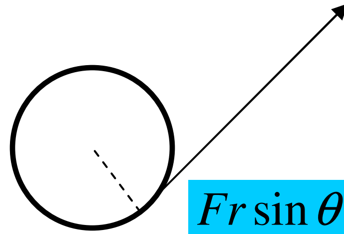
Let's first look at the ball's F.B.D



The key word here is "***rolling***". Up to this point we have always dealt with objects ***sliding*** down inclined planes. The term "rolling" tells us that FRICTION is causing the object to rotate (by applying a torque to the ball).

Example cont'





$$F_{net} = ma$$

$$mg\sin\theta - F_f = ma$$

$$F_f = mg\sin\theta - ma$$

$$Fr\sin\theta = \tau = I\alpha$$

$$F_f R = I\alpha, \quad a = \alpha r \quad I_{sphere@CM} = \tfrac{2}{3} mR^2$$

$$F_f R = \tfrac{2}{3} mR^2 \left(\frac{a}{R}\right)$$

$$F_f = \tfrac{2}{3} ma$$

$$mg\sin\theta - ma = F_f = \tfrac{2}{3} ma$$

$$mg\sin\theta = \tfrac{2}{3} ma + ma$$

$$mg\sin\theta = \tfrac{5}{3} ma$$

$$a = \frac{3g\sin\theta}{5}, \quad a = \alpha R$$

$$\alpha = \frac{3g\sin\theta}{5R}$$

Angular Momentum

Translational momentum is defined as inertia in motion. It too has an angular counterpart.

$$p = mv$$

$$L = I\omega$$

As you can see we substituted our new angular variables for the translational ones.

$$J = \Delta p$$

$$F_{net}\Delta t = \Delta p$$

$$F_{net} = \frac{\Delta p}{\Delta t}$$

$$\tau_{net} = \frac{\Delta L}{\Delta t}$$

We can look at this another way using the IMPULSE-MOMENTUM theorem

Setting Impulse equal to the change in momentum

$$\tau = F \otimes r \rightarrow Fr\sin\theta$$

$$L = p \otimes r \rightarrow pr\sin\theta \rightarrow mvr\sin\theta$$

Or we could look at this from the point of view of torque and its direct relationship with angular momentum.

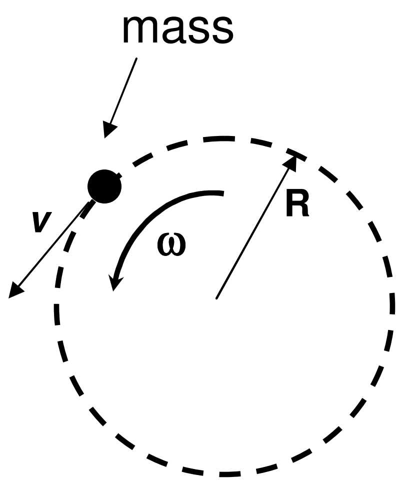
2 ways to find the angular momentum

Rotational relationship

$$L = I\omega$$

$$L = mR^2\omega$$

In the case for a mass moving in a circle.



Translational relationship

$$L = p \otimes r, \quad \theta = 90$$

$$L = mvR \quad v = R\omega$$

$$L = mR^2\omega$$

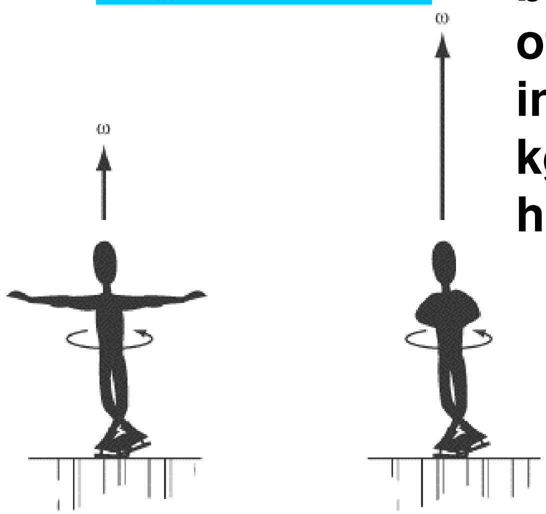
In both cases the angular momentum is the same.

Angular Momentum is also conserved

$$0 = \frac{\Delta L}{\Delta t}$$

$$0 = L - L_o$$

$$L_o = L$$



Here is what this says: IF THE NET TORQUE is equal to ZERO the CHANGE ANGULAR MOMENTUM is equal to ZERO and thus the **ANGULAR MOMENTUM is CONSERVED.**

Here is a common example. An ice skater begins a spin with his arms out. His angular velocity at the beginning of the spin is 2.0 rad/s and his moment of inertia is 6 kgm^2. As the spin proceeds he pulls in his arms decreasing his moment of inertia to 4.5 kgm^2. What is the angular velocity after pulling in his arms?

$$L_o = L$$

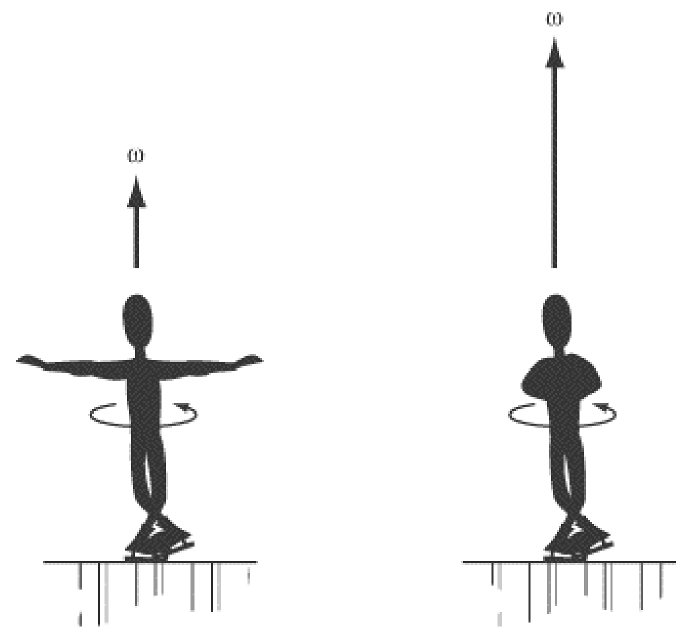
$$I\omega_o = I\omega$$

$$(6)(2) = (4.5)\omega$$

$$\omega =$$ **2.67 rad/s**

Don’t forget

Just like TORQUE, angular momentum is a cross product. That means the direction is always on a separate axis from the 2 variables you are crossing. In other words, if you cross 2 variables in the X/Y plane the cross product’s direction will be on the “Z” axis



Some interesting Calculus relationships

$$W = \int F dr = F \bullet r \rightarrow \int F_{\text{tangent}} ds, \quad ds = \text{small arc length}$$

$$\text{s} = \theta \text{r} \rightarrow \text{ds} = \text{rd}\theta$$

$$W = \int F r d\theta \rightarrow \int \tau d\theta \rightarrow \int I \alpha d\theta$$

$$\alpha = \frac{d\omega}{dt} \quad W = \int I \frac{d\omega}{dt} d\theta$$

$$\frac{d\theta}{dt} = \omega \quad W = I \int_{\omega_o}^{\omega} \omega d\omega$$

$$W = I \left(\frac{\omega^2}{2}\right)_{\omega_0}^{\omega} \rightarrow \Delta K_{Rotational}$$

More interesting calculus relationships

$$W = F \bullet \Delta r$$

$$W = \tau \bullet \Delta\theta \rightarrow \frac{W}{t} = \tau \frac{\Delta\theta}{t}, \quad \Delta \frac{\theta}{t} = \omega$$

$$P = \tau\omega$$

$$P = Fv$$