Rotational Motion I

AP Physics C
The radian

There are 2 types of pure unmixed motion:

- **Translational** - linear motion
- **Rotational** - motion involving a rotation or revolution around a fixed chosen axis (an axis which does not move).

We need a system that defines **BOTH** types of motion working together on a system. Rotational quantities are usually defined with units involving a **radian** measure.

If we take the radius of a circle and LAY IT DOWN on the circumference, it will create an angle whose arc length is equal to \( R \).

In other words, one radian angle is subtends an arc length \( \Delta s \) equal to the radius of the circle \( R \).
The radian

Half a radian would subtend an arc length equal to half the radius and 2 radians would subtend an arc length equal to two times the radius.

A general Radian Angle ($\Delta \theta$) subtends an arc length ($\Delta s$) equal to $R$. The theta in this case represents **ANGULAR DISPLACEMENT**.

$$\Delta s = R \Delta \theta$$
Angular Velocity

Since velocity is defined as the rate of change of displacement. **Angular Velocity** is defined as the rate of change of **Angular Displacement**.

\[ v = \frac{\Delta x}{\Delta t} \rightarrow \text{translational velocity} \]

\[ \bar{\omega} = \frac{\Delta \theta}{\Delta t} \rightarrow \text{rotational velocity} \]

\[ v = \frac{dx}{dt}, \quad \omega = \frac{d\theta}{dt} \]

1 revolution = \(2\pi\) radians = 360°

**NOTE:**
Translational motion tells you **THREE THINGS**
- magnitude of the motion and the units
- Axis the motion occurs on
- direction on the given axis

Example: \( \mathbf{v} = 3\mathbf{i} \)
*This tells us that the magnitude is 3 m/s, the axis is the "x" axis and the direction is in the "positive sense".*
Translation vs. Rotation

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Example: $v = 3i$
*This tells us that the magnitude is 3 m/s, the axis is the "x" axis and the direction is in the "positive sense".*

Rotational motion tells you **THREE THINGS:**
- magnitude of the motion and the units
- the PLANE in which the object rotates in
- the directional sense (counterclockwise or clockwise)

**Counterclockwise rotations are defined as having a direction of POSITIVE K motion on the "z" axis**
Rotation

Example: Unscrewing a screw or bolt

\[ = 5 \text{ rad/sec } k \]

Clockwise rotations are defined as having a direction of NEGATIVE K motion on the "z" axis

Example: Tightening a screw or bolt

\[ = -5 \text{ rad/sec } k \]
**Angular Acceleration**

\[ \bar{a} = \frac{\Delta v}{\Delta t} \rightarrow \text{translational acceleration} \]

\[ \bar{\alpha} = \frac{\Delta \omega}{\Delta t} \rightarrow \text{rotational acceleration} \]

\[ a = \frac{dv}{dt}, \alpha = \frac{d\omega}{dt} \]

Once again, following the same lines of logic. Since acceleration is defined as the rate of change of velocity. We can say the **ANGULAR ACCELERATION** is defined as the rate of change of the angular velocity.

Also, we can say that the **ANGULAR ACCELERATION** is the **TIME DERIVATIVE OF THE ANGULAR VELOCITY**.

\[ x = \int v \, dt, \quad \theta = \int \omega \, dt \]

\[ v = \int a \, dt, \quad \omega = \int \alpha \, dt \]

All the rules for integration apply as well.
Combining motions – Tangential velocity

First we take our equation for the radian measure and divide BOTH sides by a change in time.

\[ \frac{\Delta s}{\Delta t} = R \frac{\Delta \theta}{\Delta t} \]

The left side is simply the equation for linear velocity. **BUT** in this case the velocity is tangent to the circle (according to Newton’s first law). Therefore we call it **tangential velocity**.

Inspecting the right side we discover the formula for **angular velocity**.

\[ v_T = \frac{\Delta s}{\Delta t} \]

\[ \omega = \frac{\Delta \theta}{\Delta t} \]

\[ v_T = r \omega \]

Therefore, substituting the appropriate symbols we have a formula that relates translational velocity to rotational velocity.
Tangential acceleration and rotational kinematics

\[ v_t = r \omega \rightarrow \frac{v_t}{\Delta t} = \frac{r \omega}{\Delta t} \]

\[ a_t = r \alpha \]

Using the same kind of mathematical reasoning we can also define Linear tangential acceleration.

Inspecting each equation we discover that there is a DIRECT relationship between the Translational quantities and the Rotational quantities.

We can therefore RE-WRITE each translational kinematic equation and turn it into a rotational kinematic equation.

\[ v = v_o + at \rightarrow \omega = \omega_o + \alpha t \]

\[ x = x_o + v_o t + \frac{1}{2} at^2 \rightarrow \theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2 \]

\[ v^2 = v_o^2 + 2a \Delta x \rightarrow \omega^2 = \omega_o^2 + 2\alpha \Delta \theta \]
Example

A turntable capable of angularly accelerating at 12 rad/s² needs to be given an initial angular velocity if it is to rotate through a net 400 radians in 6 seconds. What must its initial angular velocity be?

\( \alpha = 12 \text{ rad} / \text{s}^2 \)

\( \Delta \theta = 400 \text{ rad} \)

\( t = 6 \text{ s} \)

\( \omega_o = ? \)

\[ \Delta \theta = \omega_o 2t + \frac{1}{2} \alpha t^2 \]

\[ 400 = \omega_o (6) + (0.5)(12)(6)^2 \]

\( \omega_o = 30.7 \text{ rad/s} \)
Rotational Kinetic Energy and Inertia

Just like massive bodies tend to resist changes in their motion (AKA - "Inertia"), rotating bodies also tend to resist changes in their motion. We call this **ROTATIONAL INERTIA**. We can determine its expression by looking at Kinetic Energy.

\[
K = \frac{1}{2} mv^2, \quad \nu_t = r \omega
\]

\[
K = \frac{1}{2} m(r\omega)^2
\]

\[
K = \frac{1}{2} mr^2 \omega^2, \quad I = \sum mr^2
\]

\[
K_{rot} = \frac{1}{2} I \omega^2
\]

We now have an expression for the rotation of a mass in terms of the radius of rotation. We call this quantity the **MOMENT OF INERTIA (I)** with units **kgm^2**.
Consider 2 masses, $m_1$ & $m_2$, rigidly connected to a bar of negligible mass. The system rotates around its CM.

This is what we would see if $m_1 = m_2$. Suppose $m_1 > m_2$.

Since it is a rigid body, the have the SAME angular velocity, $\omega$. The velocity of the center, $v_{cm}$ of mass is zero since it is rotating around it. We soon see that the TANGENTIAL SPEEDS are NOT EQUAL due to different radii.

$$v_t = r\omega$$
Moment of Inertia, $I$

Since both masses are moving they have kinetic energy or rotational kinetic in this case.

$$K = K_1 + K_2$$

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2, \quad v_t = r \omega$$

$$K = \frac{1}{2} m_1 (r_1^2 \omega^2) + \frac{1}{2} m_2 (r_2^2 \omega^2)$$

$$K = \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \omega^2$$

$$K = \frac{1}{2} \left( \sum_{i=1}^{N} m_i r_i^2 \right) \omega^2 \rightarrow K = \frac{1}{2} I \omega^2$$

So this example clearly illustrates the idea behind the SUMMATION in the moment of inertia equation.
Example

A very common problem is to find the velocity of a ball rolling down an inclined plane. It is important to realize that you cannot work out this problem the way you used to. In the past, everything was SLIDING. Now the object is rolling and thus has MORE energy than normal. So let’s assume the ball is like a thin spherical shell and was released from a position 5 m above the ground. Calculate the velocity at the bottom of the incline.

\[ E_{\text{before}} = E_{\text{after}} \]
\[ U_g = K_T + K_R \]
\[ mgh = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2 \]
\[ v = R \omega \quad I_{\text{sphere @ cm}} = \frac{2}{3} mR^2 \]
\[ mgh = \frac{1}{2} mv^2 + \frac{1}{2} \left( \frac{2}{3} mR^2 \right) \left( \frac{v^2}{R^2} \right) \]
\[ gh = \frac{1}{2} v^2 + \frac{1}{3} v^2 \]
\[ v = \sqrt{6/5 \cdot gh} = \sqrt{6/5 \cdot (9.8)\cdot(5)} = 7.67 \text{ m/s} \]

If you HAD NOT included the rotational kinetic energy, you see the answer is very much different.

\[ E_{\text{before}} = E_{\text{after}} \]
\[ U_g = K_T \]
\[ mgh = \frac{1}{2} mv^2 \]
\[ mgh = \frac{1}{2} mv^2 \]
\[ gh = \frac{1}{2} v^2 \]
\[ v = \sqrt{2 \cdot gh} = \sqrt{2 \cdot (9.8)\cdot(5)} = 9.90 \text{ m/s} \]
Example: Moment of Inertia, $I$

Let's use this equation to analyze the motion of a 4-m long bar with negligible mass and two equal masses (3-kg) on the end rotating around a specified axis.

$$I = \sum mr^2$$

**EXAMPLE #1** - The moment of Inertia when they are rotating around the center of their rod.

$$I = \sum mr^2 = mr^2 + mr^2$$

$$I = (3)(2)^2 + (3)(2)^2 = 24 \text{ kgm}^2$$

**EXAMPLE #2** - The moment if Inertia rotating at one end of the rod

$$I = \sum mr^2 = mr^2 + mr^2$$

$$I = (3)(0)^2 + (3)(4)^2 = 48 \text{ kgm}^2$$
Example cont’

Now let’s calculate the moment of Inertia rotating at a point 2 meters from one end of the rod.

\[
I = \sum mr^2 = mr^2 + mr^2
\]

\[
I = (3)(2)^2 + (3)(6)^2 = 120 \text{ kgm}^2
\]

As you can see, the FARTHER the axis of rotation is from the center of mass, the moment of inertia increases. We need an expression that will help us determine the moment of inertia when this situation arises.
Parallel Axis Theorem

This theorem will allow us to calculate the moment of inertia of any rotating body around any axis, provided we know the moment of inertia about the center of mass.

\[ I_p = I_{cm} + M d^2 \]

It basically states that the Moment of Inertia (\(I_p\)) around any axis "P" is equal to the known moment of inertia (\(I_{cm}\)) about some center of mass plus \(M\) (the total mass of the system) times the square of "d" (the distance between the two parallel axes).

Using the prior example let’s use the parallel axis theorem to calculate the moment of inertia when it is rotating around one end and 2m from a fixed axis.
Exam – Parallel Axis Theorem

\[ I_p = I_{cm} +Md^2 \]

\[ I_p = (24) + (6)(2)^2 = 48 \text{ kgm}^2 \]

\[ I_p = I_{cm} +Md^2 \]

\[ I_p = (24) + (6)(4)^2 = 120 \text{ kgm}^2 \]
Continuous Masses

The earlier equation, $I = \sum \sum \sum mr^2$, worked fine for what is called POINT masses. But what about more continuous masses like disks, rods, or sphere where the mass is extended over a volume or area. In this case, calculus is needed.

This suggests that we will take small discrete amounts of mass and add them up over a set of limits. Indeed, that is what we will do. Let’s look at a few example we “MIGHT” encounter. Consider a solid rod rotating about its CM.

Will, $I = \sum mr^2$, be the equation for a rod?
We begin by using the same technique used to derive the center of mass of a continuous body.

\[ I = \sum mr^2 \rightarrow I = \int r^2 \, dm \]

**Macro** → \( \lambda = \frac{M}{L} \)

**Micro** → \( \lambda = \frac{dm}{dx} = \frac{dm}{dr} \)

\[ dm = \lambda dr \]

\[ I = \int r^2 \, dm \rightarrow \int_{-L/2}^{L/2} r^2 (\lambda) \, dr \]

\[ I = \lambda \int_{-L/2}^{L/2} r^2 \, dr \rightarrow \frac{M}{L} \left| \frac{r^3}{3} \right|_{-L/2}^{L/2} \]

\[ I = \frac{M}{L} \left( \frac{L^3}{8} - \frac{-L^3}{8} \right) \rightarrow \frac{M}{L} \left( \frac{L^3}{24} + \frac{L^3}{24} \right) = I_{rod \ @ \ cm} = \frac{ML^2}{12} \]
Your turn

What if the rod were rotating on one of its ENDS?

\[ I = \sum mr^2 \rightarrow I = \int r^2 dm \]

Macro \rightarrow \lambda = \frac{M}{L}

Micro \rightarrow \lambda = \frac{dm}{dx} = \frac{dm}{dr} \quad dm = \lambda dr

\[ I = \int r^2 dm \rightarrow \int_0^L r^2 (\lambda) dr \]

\[ I = \lambda \int_0^L r^2 dr \rightarrow \frac{M}{L} \left| \frac{r^3}{3} \right|_0^L \]

\[ I = \frac{M}{L} \left( \frac{L^3}{3} - 0 \right) = I_{\text{rod @ end}} = \frac{ML^2}{3} \]

As you can see you get a completely different expression depending on HOW the body is rotating.
**The disk**

\[ I = \sum mr^2 \rightarrow I = \int r^2 dm \]

\[ \text{Macro} \rightarrow \sigma = \frac{M}{A} = \frac{M}{\pi R^2} \]

\[ \text{Micro} \rightarrow \sigma = \frac{dm}{dA} = \frac{dm}{2\pi dx \text{ (or } dr)} \]

\[ dm = \sigma 2\pi r dr \]

\[ I = \int r^2 dm \rightarrow \int_0^R r^2 (\sigma 2\pi r) dr \]

\[ I = 2\pi \sigma \int_0^R r^3 dr \rightarrow \frac{2\pi M}{\pi R^2} \mid \frac{r^4}{4} \mid_0^R \]

\[ I = \frac{2M}{R^2} \left( \frac{R^4}{4} - 0 \right) = I_{\text{disk @ cm}} = \frac{MR^2}{2} \]
The bottom line..

Will you be asked to derive the moment of inertia of an object? Possibly! Fortunately, most of the time the moment of inertia is given within the free response question.

Consult the file (on the notes page) called Moments of Inertia to view common expressions for “I” for various shapes and rotational situations.