Rotational Motion I

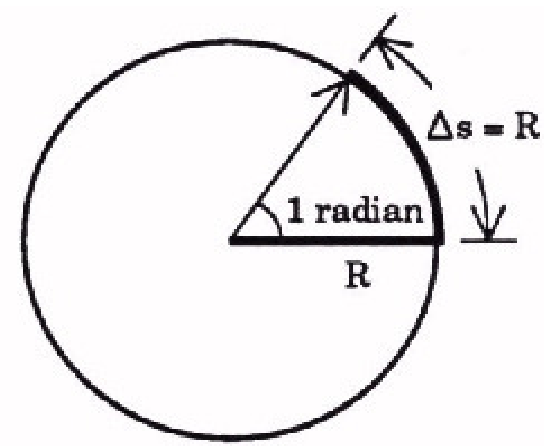
AP Physics C

The radian

There are 2 types of pure unmixed motion:

- **Translational** - linear motion
- **Rotational** - motion involving a rotation or revolution around a fixed chosen axis(***an axis which does not move***).

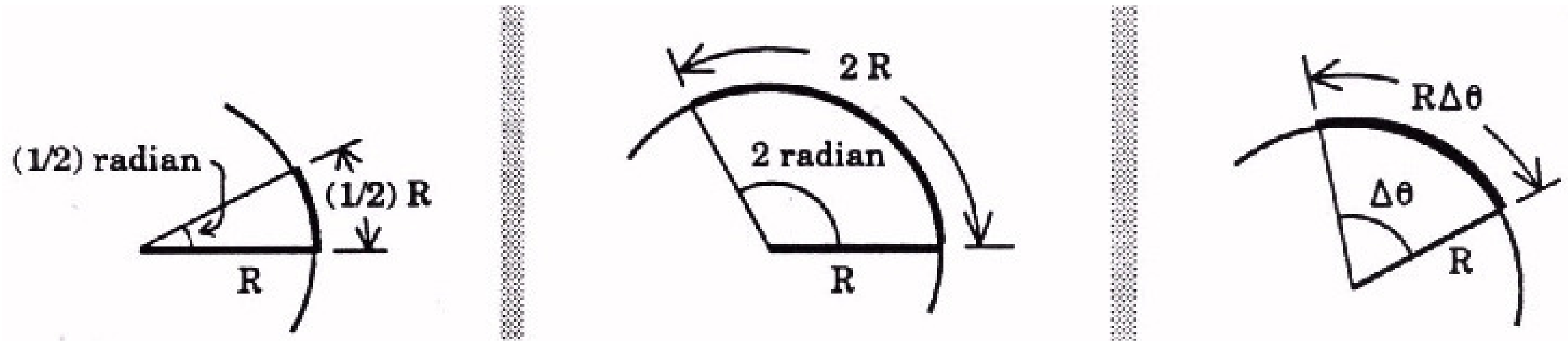
We need a system that defines **BOTH** types of motion working together on a system. Rotational quantities are usually defined with units involving a ***radian*** measure.



If we take the radius of a circle and **LAY IT DOWN** on the circumference, it will create an angle whose arc length is equal to **R**.

In other words, one radian angle is subtends an arc length ***Δs*** equal to the radius of the circle (***R***)

The radian



Half a radian would subtend an arc length equal to half the radius and 2 radians would subtend an arc length equal to two times the radius.

A general Radian Angle ($\Delta\theta$) subtends an arc length (Δs) equal to R. The theta in this case represents **ANGULAR DISPLACEMENT.**

$$\Delta s = R\Delta\theta$$

Angular Velocity

$$\bar{v} = \frac{\Delta x}{\Delta t} \rightarrow translational\ velocity$$

$$\bar{\varpi} = \frac{\Delta \theta}{\Delta t} \rightarrow rotational\ velocity$$

$$v = \frac{dx}{dt}, \omega = \frac{d\theta}{dt}$$

Since velocity is defined as the rate of change of displacement. **ANGULAR VELOCITY** is defined as the rate of change of **ANGULAR DISPLACEMENT**.

$$1\,revolution = 2\pi\,radians = 360^{\circ}$$

NOTE:

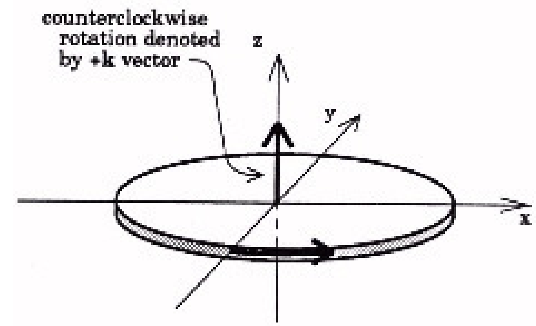
Translational motion tells you **THREE THINGS**

- magnitude of the motion and the units
- Axis the motion occurs on
- direction on the given axis

Example: ***v =3i***

This tells us that the magnitude is 3 m/s, the axis is the "x" axis and the direction is in the "positive sense".

Translation vs. Rotation



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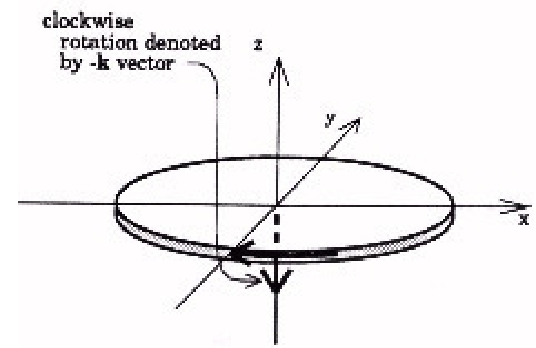
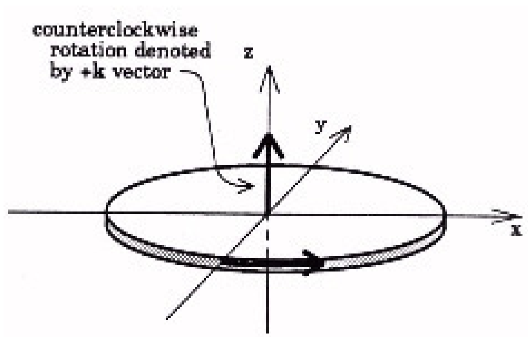


Rotational motion tells you **THREE THINGS**:
- magnitude of the motion and the units
- the PLANE in which the object rotates in
- the directional sense (counterclockwise or clockwise)

Counterclockwise rotations are defined as having a direction of POSITVE K motion on the "z" axis

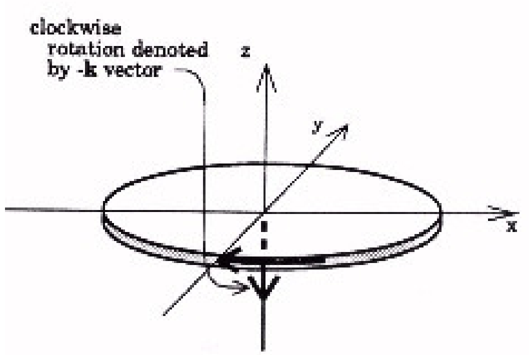
Rotation



Example: Unscrewing a screw or bolt

= 5 rad/sec ***k***

Clockwise rotations are defined as having a direction of NEGATIVE K motion on the "z" axis



Example: Tightening a screw or bolt

= -5 rad/sec ***k***

Angular Acceleration

$$\bar{a} = \frac{\Delta v}{\Delta t} \rightarrow translational\ acceleration$$

$$\bar{\alpha} = \frac{\Delta \omega}{\Delta t} \rightarrow rotational\ acceleration$$

$$a = \frac{dv}{dt}, \alpha = \frac{d\omega}{dt}$$

$$x = \int v\,dt, \quad \theta = \int \omega\,dt$$

$$v = \int a\,dt, \quad \omega = \int \alpha\,dt$$

Once again, following the same lines of logic. Since acceleration is defined as the rate of change of velocity. We can say the **ANGULAR ACCELERATION is defined as the rate of change of the angular velocity.**

Also, we can say that the **ANGULAR ACCELERATION is the TIME DERIVATIVE OF THE ANGULAR VELOCITY**.

All the rules for integration apply as well.

Combining motions –Tangential velocity

$$\Delta s = R\Delta\theta$$

$$\frac{\Delta s}{\Delta t} = R\frac{\Delta\theta}{\Delta t}$$

$$v_T = \frac{\Delta s}{\Delta t}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$

$$v_T = r\omega$$

First we take our equation for the radian measure and divide BOTH sides by a change in time.

The left side is simply the equation for **LINEAR** velocity. **BUT** in this case the velocity is **TANGENT** to the circle (*according to Newton's first law*). Therefore we call it **TANGENTIAL VELOCITY**.

Inspecting the right side we discover the formula for **ANGULAR VELOCITY.**

Therefore, substituting the appropriate symbols we have a formula that relates Translational velocity to Rotational velocity.

Tangential acceleration and rotational kinematics

$$v_t = r\omega \rightarrow \frac{v_t}{\Delta t} = \frac{r\omega}{\Delta t}$$

$$a_t = r\alpha$$

Using the same kind of mathematical reasoning we can also define Linear tangential acceleration.

Inspecting each equation we discover that there is a DIRECT relationship between the Translational quantities and the Rotational quantities.

We can therefore **RE-WRITE** each translational kinematic equation and turn it into a rotational kinematic equation.

$$v = v_o + at \rightarrow \quad \omega = \omega_o + \alpha t$$

$$x = x_o + v_o t + \tfrac{1}{2}at^2 \rightarrow \quad \theta = \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2$$

$$v^2 = v_o^2 + 2a\Delta x \rightarrow \quad \omega^2 = \omega_o^2 + 2\alpha\Delta\theta$$

Example

A turntable capable of angularly accelerating at 12 rad/s^2 needs to be given an initial angular velocity if it is to rotate through a net 400 radians in 6 seconds. What must its initial angular velocity be?

$$\alpha = 12\ rad/s^2$$

$$\Delta\theta = 400\ rad$$

$$t = 6\ s$$

$$\omega_o = ?$$

$$\Delta\theta = \omega_o 2t + \tfrac{1}{2}\alpha t^2$$

$$400 = \omega_o(6) + (0.5)(12)(6)^2$$

$\omega_o =$ **30.7 rad/s**

Rotational Kinetic Energy and Inertia

Just like massive bodies tend to resist changes in their motion (AKA - "Inertia") . Rotating bodies also tend to resist changes in their motion. We call this **ROTATIONAL INERTIA**. We can determine its expression by looking at Kinetic Energy.

$$K = \tfrac{1}{2} m v^2, \quad v_t = r\omega$$

$$K = \tfrac{1}{2} m (r\omega)^2$$

$$K = \tfrac{1}{2} \boxed{mr^2} \omega^2, \quad I = \sum mr^2$$

$$K_{rot} = \tfrac{1}{2} I \omega^2$$

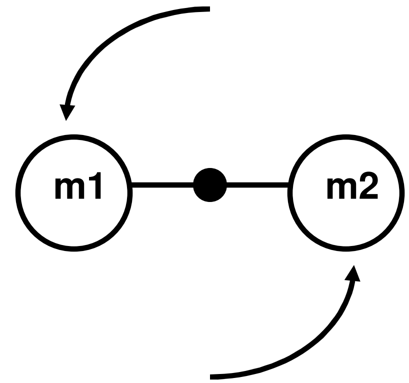
We now have an expression for the rotation of a mass in terms of the radius of rotation.
We call this quantity the **MOMENT OF INERTIA (I)** with units **kgm^2**

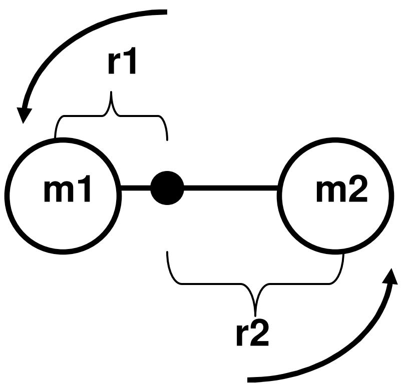
Moment of Inertia, I

$$I = \sum mr^2$$

Consider 2 masses, m1 & m2, rigidly connected to a bar of negligible mass. The system rotates around its CM.

This is what we would see if m1 = m2.
Suppose m1>m2.





Since it is a rigid body, the have the SAME angular velocity, ω. The velocity of the center, v_{cm} of mass is zero since it is rotating around it. We soon see that the TANGENTIAL SPEEDS are NOT EQUAL due to different radii.

$$v_t = r\omega$$

Moment of Inertia, I

Since both masses are moving they have kinetic energy or rotational kinetic in this case.

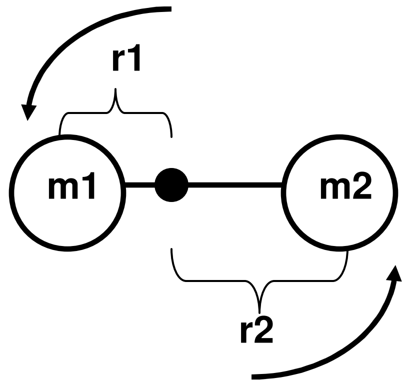
$$K = K_1 + K_2$$

$$K = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2, \quad v_t = r\omega$$

$$K = \tfrac{1}{2} m_1 (r_1^2 \omega^2) + \tfrac{1}{2} m_2 (r_2^2 \omega^2)$$

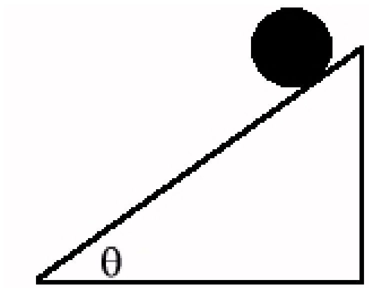
$$K = \tfrac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \omega^2$$

$$K = \tfrac{1}{2} \left(\sum_{i=1}^{N} m_i r_i^2\right) \omega^2 \rightarrow K = \tfrac{1}{2} I \omega^2$$



So this example clearly illustrates the idea behind the SUMMATION in the moment of inertia equation.

Example



A very common problem is to find the velocity of a ball rolling down an inclined plane. It is important to realize that you cannot work out this problem they way you used to. In the past, everything was SLIDING. Now the object is rolling and thus has MORE energy than normal. So let's assume the ball is like a thin spherical shell and was released from a position 5 m above the ground. Calculate the velocity at the bottom of the incline.

$$E_{before} = E_{after}$$

$$U_g = K_T + K_R$$

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

$$v = R\omega \quad I_{sphere@cm} = \tfrac{2}{3}mR^2$$

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}(\tfrac{2}{3}mR^2)(\frac{v^2}{R^2})$$

$$gh = \tfrac{1}{2}v^2 + \tfrac{1}{3}v^2$$

$$v = \sqrt{\tfrac{6}{5}gh} = \sqrt{\tfrac{6}{5}(9.8)(5)} =$$ **7.67 m/s**

$$E_{before} = E_{after}$$

$$U_g = K_T$$

$$mgh = \tfrac{1}{2}mv^2$$

$$mgh = \tfrac{1}{2}mv^2$$

$$gh = \tfrac{1}{2}v^2$$

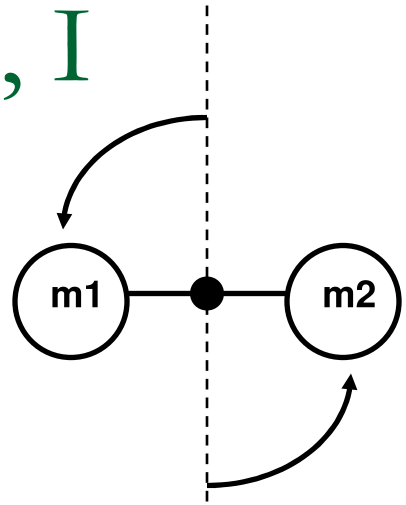
$$v = \sqrt{2gh} = \sqrt{2(9.8)(5)} = 9.90\ m/s$$

If you HAD NOT included the rotational kinetic energy, you see the answer is very much different.

Example: Moment of Inertia, I

$$I = \sum mr^2$$

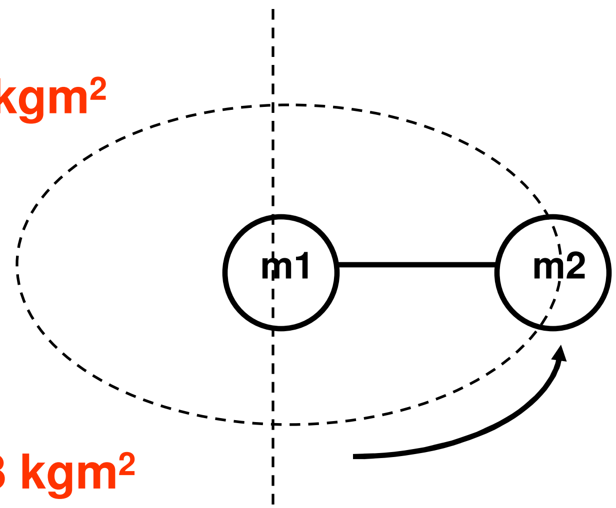
Let's use this equation to analyze the motion of a 4-m long bar with negligible mass and two equal masses(3-kg) on the end rotating around a specified axis.



EXAMPLE #1 -The moment of Inertia when they are rotating around the center of their rod.

$$I = \sum mr^2 = mr^2 + mr^2$$

$$I = (3)(2)^2 + (3)(2)^2 =$$ **24 kgm²**



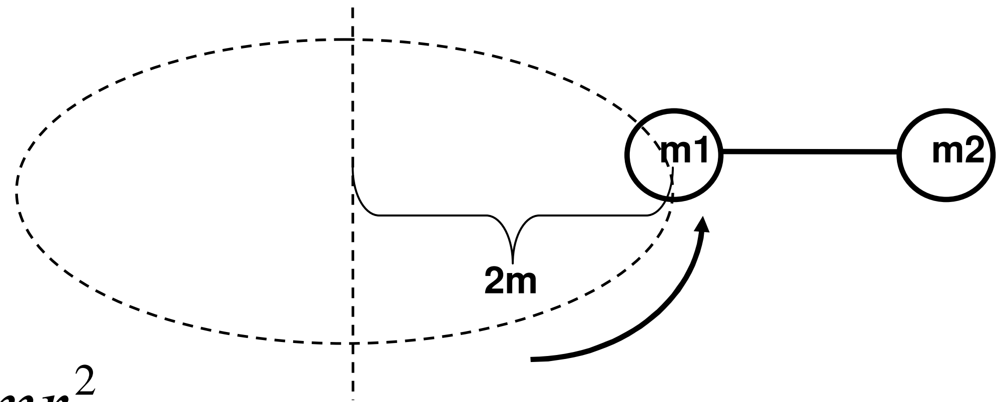
EXAMPLE #2-The moment if Inertia rotating at one end of the rod

$$I = \sum mr^2 = mr^2 + mr^2$$

$$I = (3)(0)^2 + (3)(4)^2 =$$ **48 kgm²**

Example cont'

Now let's calculate the moment of Inertia rotating at a point 2 meters from one end of the rod.



$$I = \sum mr^2 = mr^2 + mr^2$$

$$I = (3)(2)^2 + (3)(6)^2 =$$ **120 kgm^2**

As you can see, the FARTHER the axis of rotation is from the center of mass, the moment of inertia increases. We need an expression that will help us determine the moment of inertia when this situation arises.

Parallel Axis Theorem

This theorem will allow us to calculate the moment of inertia of any rotating body around any axis, provided we know the moment of inertia about the center of mass.

$$I_p = I_{cm} + Md^2$$

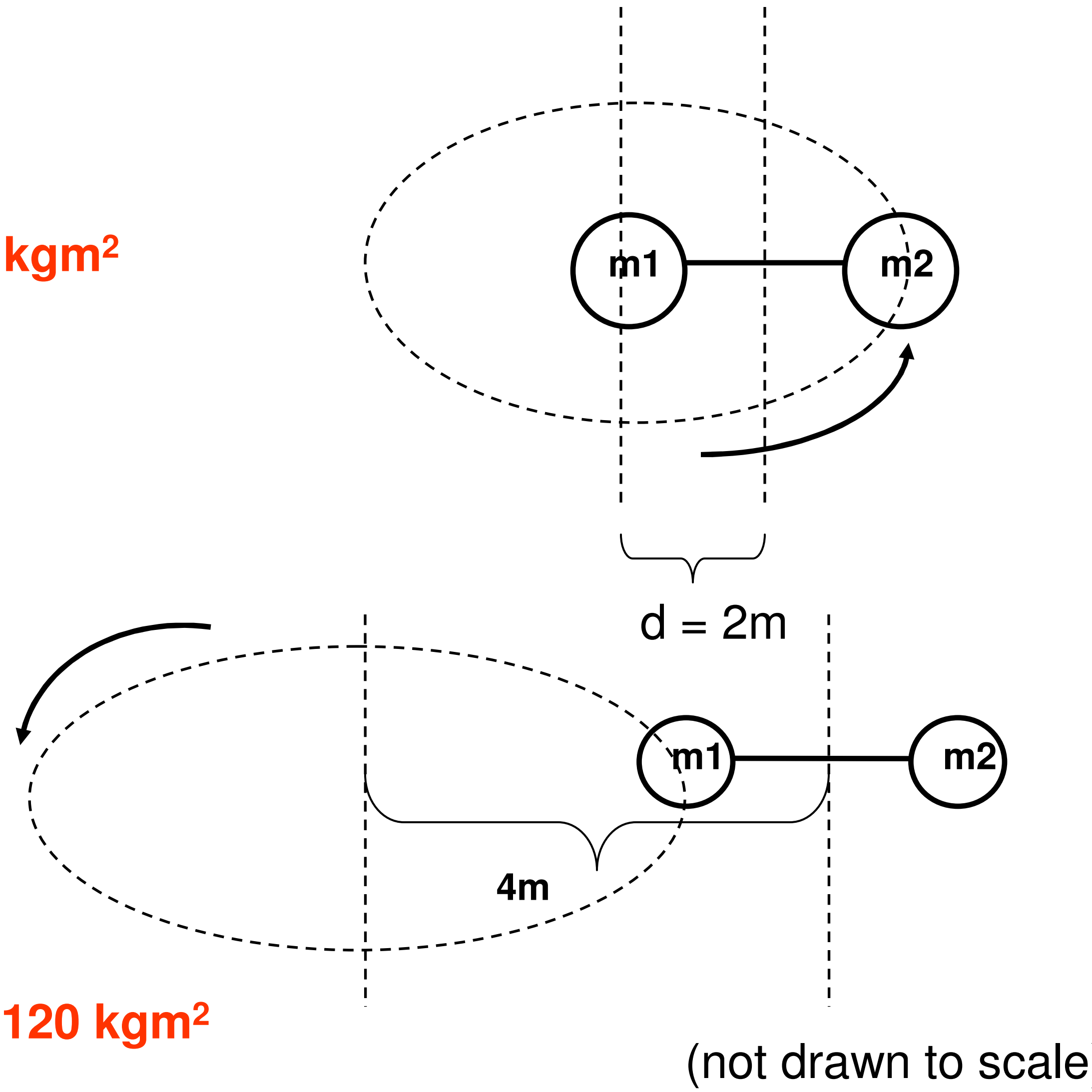
It basically states that the Moment of Inertia **(I_p)** around any axis "P" is equal to the known moment of inertia **(I_{cm})** about some center of mass plus **M** (the total mass of the system) times the square of "**d**" (the distance between the two parallel axes)

Using the prior example let's use the parallel axis theorem to calculate the moment of inertia when it is rotating around one end and 2m from a fixed axis.

Exam – Parallel Axis Theorem

$$I_p = I_{cm} + Md^2$$

$$I_p = (24) + (6)(2)^2 =$$ **48 kgm²**



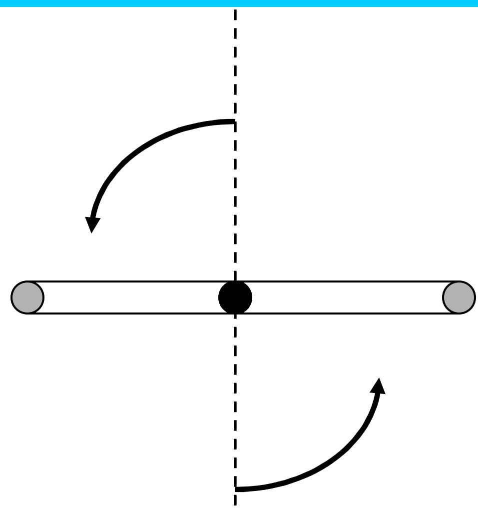
$$I_p = I_{cm} + Md^2$$

$$I_p = (24) + (6)(4)^2 =$$ **120 kgm²**

Continuous Masses

The earlier equation, **$I = \Sigma mr^2$**, worked fine for what is called POINT masses. But what about more continuous masses like disks, rods, or sphere where the mass is extended over a volume or area. In this case, calculus is needed.

$$I = \sum mr^2 \rightarrow I = \int r^2 dm$$

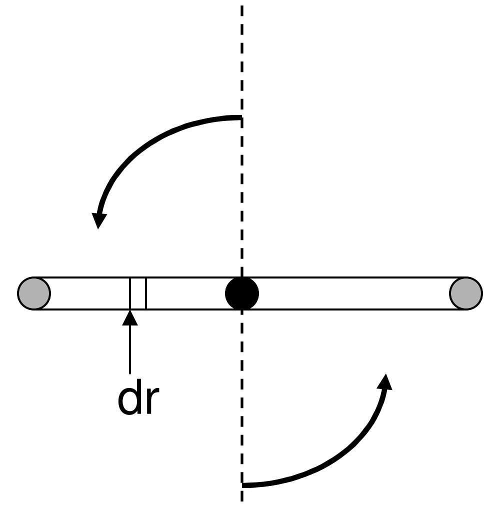


This suggests that we will take small discrete amounts of mass and add them up over a set of limits. Indeed, that is what we will do. Let's look at a few example we "MIGHT" encounter. Consider a solid rod rotating about its CM.

Will, **$I = \Sigma mr^2$, be the equation for a rod?**

The rod

We begin by using the same technique used to derive the center of mass of a continuous body.



$$I = \sum mr^2 \rightarrow I = \int r^2 dm$$

$$Macro \rightarrow \lambda = \frac{M}{L}$$

$$Micro \rightarrow \lambda = \frac{dm}{dx} = \frac{dm}{dr} \qquad dm = \lambda dr$$

The CM acts as the origin in the case of determining the limits.

$$I = \int r^2 dm \rightarrow \int_{-L/2}^{L/2} r^2 (\lambda) dr$$

$$I = \lambda \int_{-L/2}^{L/2} r^2 dr \rightarrow \frac{M}{L} \left| \frac{r^3}{3} \right|_{-L/2}^{L/2}$$

$$I = \frac{M}{L}\left(\frac{L^3/8}{3} - \frac{-L^3/8}{3}\right) \rightarrow \frac{M}{L}\left(\frac{L^3}{24} + \frac{L^3}{24}\right) = \boxed{I_{rod\,@\,cm} = \frac{ML^2}{12}}$$

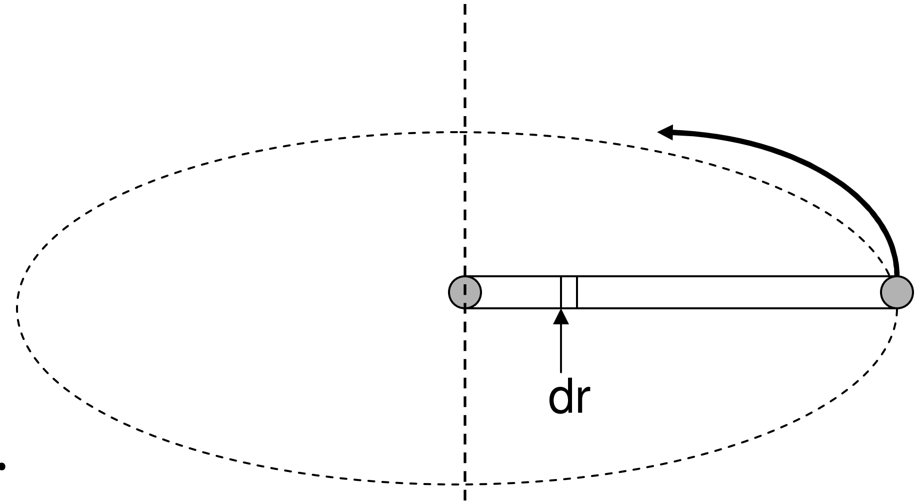
Your turn

What if the rod were rotating on one of its ENDS?

$$I = \sum mr^2 \rightarrow I = \int r^2 dm$$

$$Macro \rightarrow \lambda = \frac{M}{L}$$

$$Micro \rightarrow \lambda = \frac{dm}{dx} = \frac{dm}{dr} \qquad dm = \lambda dr$$



$$I = \int r^2 dm \rightarrow \int_0^L r^2 (\lambda) dr$$

$$I = \lambda \int_0^L r^2 dr \rightarrow \frac{M}{L} \mid \frac{r^3}{3} \mid_0^L$$

$$I = \frac{M}{L}(\frac{L^3}{3} - 0) = \quad I_{rod\,@\,end} = \frac{ML^2}{3}$$

As you can see you get a completely different expression depending on HOW the body is rotating.

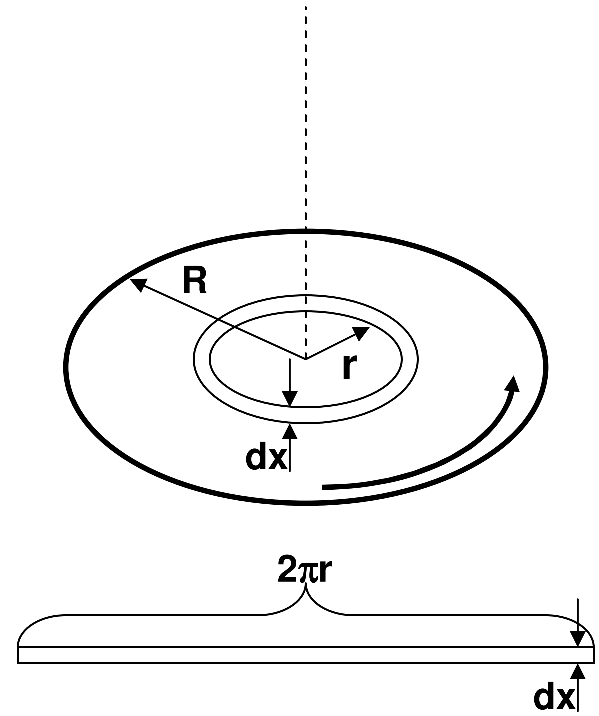
The disk

$$\boxed{I = \sum mr^2 \rightarrow I = \int r^2 dm}$$

$$Macro \rightarrow \sigma = \frac{M}{A} = \frac{M}{\pi R^2}$$

$$Micro \rightarrow \sigma = \frac{dm}{dA} = \frac{dm}{2\pi r dx (or\ dr)}$$

$$dm = \sigma 2\pi r dr$$



$$I = \int r^2 dm \rightarrow \int_0^R r^2 (\sigma 2\pi r) dr$$

$$I = 2\pi\sigma \int_0^R r^3 dr \rightarrow \frac{2\pi M}{\pi R^2} \left| \frac{r^4}{4} \right|_0^R$$

$$I = \frac{2M}{R^2}\left(\frac{R^4}{4} - 0\right) =$$

$$\boxed{I_{disk @ cm} = \frac{MR^2}{2}}$$

The bottom line..

Will you be asked to derive the moment of inertia of an object? Possibly! Fortunately, most of the time the moment of inertia is given within the free response question.

Consult the file (on the notes page) called Moments of Inertia to view common expressions for "I" for various shapes and rotational situations.