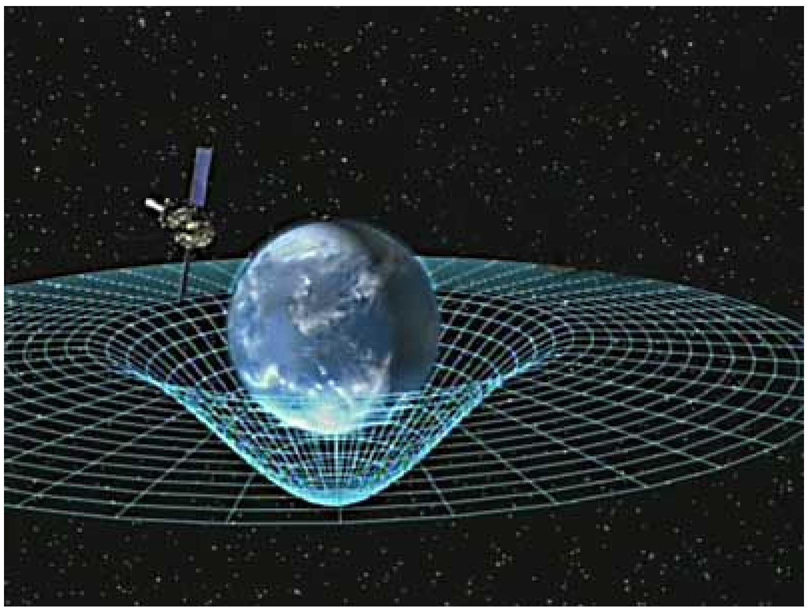
Gravitation

AP Physics C

Newton's Law of Gravitation

What causes YOU to be pulled down? THE EARTH….or more specifically…the EARTH'S MASS. Anything that has MASS has a gravitational pull towards it.



$$F_g \alpha \; Mm$$

What the proportionality above is saying is that for there to be a FORCE DUE TO GRAVITY on something there must be at least 2 masses involved, where one is larger than the other.

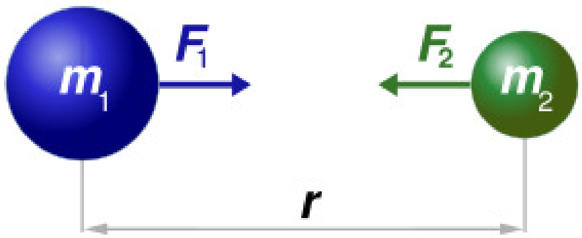
N.L.o.G.

As you move AWAY from the earth, your DISTANCE increases and your FORCE DUE TO GRAVITY decrease. This is a special INVERSE relationship called an Inverse-Square.

$$F_g \; \alpha \frac{1}{r^2}$$

The “r” stands for SEPARATION DISTANCE and is the distance between the CENTERS OF MASS of the 2 objects. We us the symbol “r” as it symbolizes the radius. Gravitation is closely related to circular motion as you will discover later.

N.L.o.G – Putting it all together



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

$$F_g \; \alpha \frac{m_1 m_2}{r^2}$$

$G =$ constant of proportionality

G = Universal Gravitational Constant

$$G = 6.67x10^{-27} \; {Nm^2}/{kg^2}$$

$$F_g = G\frac{m_1 m_2}{r^2}$$

$F_g = mg \rightarrow$ Use this when you are on the earth

$F_g = G\frac{m_1 m_2}{r^2} \rightarrow$ Use this when you are LEAVING the earth

Try this!

Let’s set the 2 equations equal to each other since they BOTH represent your weight or force due to gravity

$$F_g = mg \rightarrow \text{Use this when you are on the earth}$$

$$F_g = G\frac{m_1 m_2}{r^2} \rightarrow \text{Use this when you are LEAVING the earth}$$

$$mg = G\frac{Mm}{r^2}$$

$$g = G\frac{M}{r^2}$$

$$M = \text{Mass of the Earth} = 5.97x10^{24} - kg$$

$$r = \text{radius of the Earth} = 6.37x10^{6} - m$$

SOLVE FOR g!

$$g = \frac{(6.67x10^{-27})(5.97x10^{24})}{(6.37x10^{6})^2} = 9.81 m/s^2$$

How did Newton figure this out?

Newton knew that the force on a falling apple (due to Earth) is in direct proportion to the acceleration of that apple. He also knew that the force on the moon is in direct proportion to the acceleration of the moon, ALSO due to Earth

$$F_{apple} \alpha \, a_{apple}$$

$$F_{moom} \alpha \, a_{moon}$$

$$F_{apple} \alpha \, \frac{1}{r^{n}_{apple}}$$

$$F_{moon} \alpha \, \frac{1}{r^{n}_{moon}}$$

Newton also surmised that that SAME force was inversely proportional to the distance from the center of Earth. The problem was that he wasn't exactly sure what the exponent was.

How did Newton figure this out?

$$\frac{a_{apple}}{a_{moon}} \alpha \frac{\frac{1}{r^{n}_{apple}}}{\frac{1}{r^{n}_{moon}}}$$

$$\frac{a_{apple}}{a_{moon}} \alpha \left(\frac{r_{moon}}{r_{apple}}\right)^{n}$$

$$\frac{9.8}{0.002722} \alpha \frac{240,000}{4000}$$

$$3600 \alpha 60^{n}$$

$$n = 2$$

Since both the acceleration and distance were set up as proportionalities with the force, he decided to set up a ratio.

Newton knew that the acceleration of the apple was 9.8 and that the approximate distance was 4000 miles to the center of Earth.

Newton also knew the distance and acceleration of the Moon as it orbits Earth centripetally. It was the outcome of this ratio that led him to the exponent of “2”. Therefore creating an inverse square relationship.

Newton's Law of Gravitation (in more detail)

To make the expression more mathematically acceptable we also look at this formula this way:

$$F_g = -G\frac{m_1 m_2}{r^2}\vec{r}$$

The NEW "r" that you see is simply a unit vector like I,j, & k-hat. A unit vector, remember, tells you the direction the force is going. In this case it means that it is between the two bodies is RADIAL in nature. The NEGATIVE SIGN is meant to denote that a force produces "bound" orbits. It is only used when you are sure you need it relative to whatever reference frame you are using .....SO BE CAREFUL! It may be wise to use this expression to find magnitudes only.

Example

What is the gravitational force between the earth and a 100 kg man standing on the earth's surface?

$$M = \text{Mass of the Earth} = 5.97x10^{24} - kg$$

$$r = \text{radius of the Earth} = 6.37x10^{6} - m$$

$$F_g = G\frac{m_{man}M_{Earth}}{r^2} = 6.67x10^{-11}\frac{(100)(5.97x10^{24})}{(6.37x10^{6})^2} =$$ **9.81 x 10² N**

Because the force near the surface of Earth is constant, we can define this force easier by realizing that this force of gravitation is in direct proportional to the man's mass. A constant of proportionality must drive this relationship.

$$F_g \alpha\, m_{man} \rightarrow F_g = m_{man} g$$

$$9.81x10^2 = 100g$$

$$g = 9.8\, m/s/s$$

We see that this constant is in fact the gravitational acceleration located near the Earth's surface.

Example

How far from the earth's surface must an astronaut in space be if she is to feel a gravitational acceleration that is half what she would feel on the earth's surface?

$$g = G\frac{M}{(r + r_{earth})^2} \rightarrow r = \sqrt{\frac{GM_{Earth}}{g}} - r_{Earth}$$

$$mg = G\frac{Mm}{r^2} \qquad r = \sqrt{\frac{(6.67x10^{-11})(5.97x10^{24})}{4.9}} - 6.37x10^6 =$$ **2.64x10⁶ m**

$$g = G\frac{M}{r^2}$$

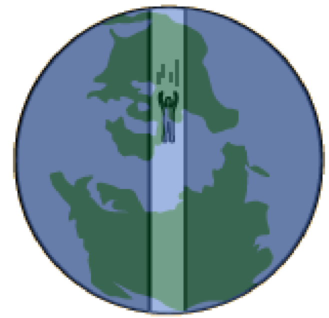
M = Mass of the Earth $= 5.97x10^{24} - kg$

r = radius of the Earth $= 6.37x10^6 - m$

This value is four tenths the radius of Earth.

A couple of things to consider about Earth

- **You can treat the earth as a point mass with its mass being at the center if an object is on its surface**
- **The earth is actually not uniform**
- **The earth is not a sphere**
- **The earth is rotating**

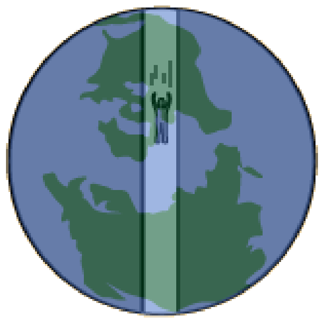


Let's assume the earth is a uniform sphere.

What would happen to a mass (man) that is dropped down a hole that goes completely through the earth?

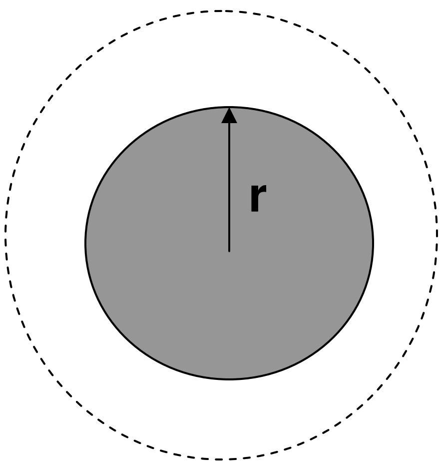
Digging a hole at the Forbidden City in Beijing will cause you to end up somewhere in Argentina. But don't be surprised if you dig somewhere else and water starts to pour in!

Digging a hole



When you jump down and are at a radius "r" from the center, the portion of Earth that lies OUTSIDE a sphere a radius "r" does NOT produce a NET gravitational force on you!

The portion that lies INSIDE the sphere does. This implies that as you fall the "sphere" changes in volume, mass, and density (due to different types of rocks)



$$\rho = \frac{M}{V}, V_{sphere} = \frac{4}{3}\pi r^3 \rightarrow M_{inside} = \rho\frac{4\pi r^3}{3}$$

$$F_g = G\frac{Mm}{r^2} \rightarrow F_g = \frac{G4\pi m\rho}{3}r \quad k = \frac{G4\pi m\rho}{3}$$

$$F_g = -kr$$

This tells us that your "weight" actually DECREASES as you approach the center of Earth from within the INSIDE of the sphere and that it behaves like Hook's Law. YOU WILL OSCILLATE.

Energy Considerations

$$F_g = -G\frac{m_1 m_2}{r^2}\vec{r}$$

$$W = \int_R^{\infty} F(r)dr$$

$$W = \int_R^{\infty} G\frac{mM}{r^2}dr$$

$$W = GmM\int_R^{\infty}\frac{1}{r^2}dr = GmM\int_R^{\infty} r^{-2}$$

$$W = U_g = -\frac{GmM}{r}$$

Work is the integral of a Force function with respect to displacement.

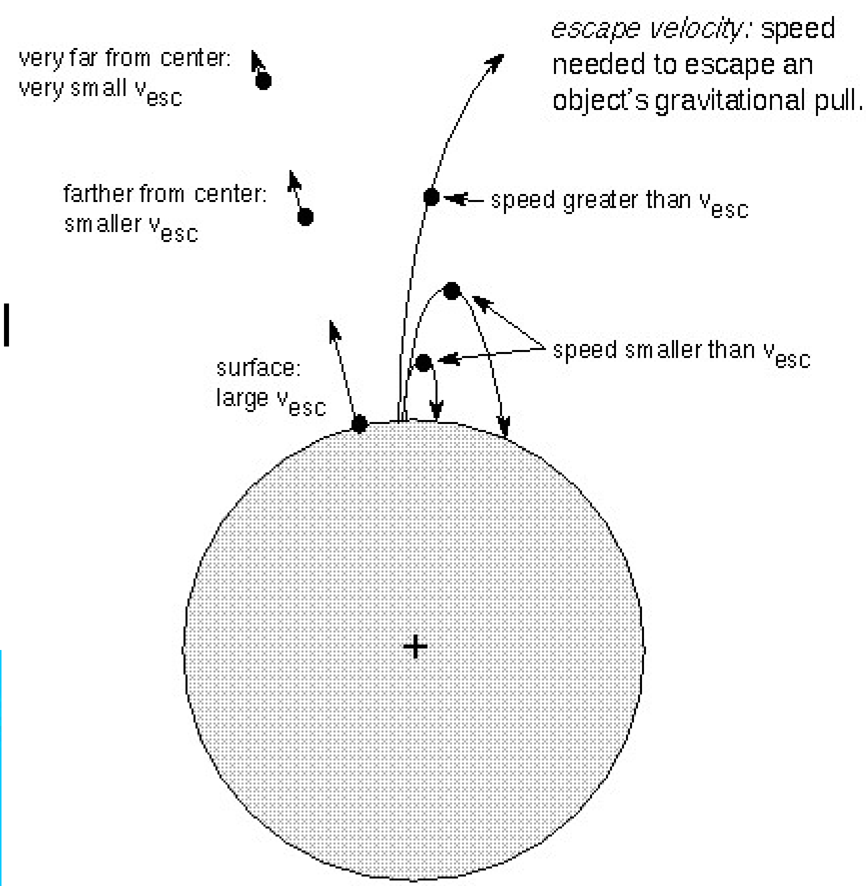
Putting in the basic expression for gravitational force

Pulling out the constants and bringing the denominator to the numerator.

The negative sign should not surprise you as we already knew that Work was equal to the negative change in "U" or mgh.

Escape Speed

Consider a rocket leaving the earth. It usually goes up, slows down, and then returns to earth. There exists an initial minimum speed that when reached the rockets will continue on forever. **Let's use conservation of energy to analyze this situation!**



$$U_g = -\frac{GmM}{r}, K = \frac{1}{2}mv^2$$

$$U_g + K = 0, U_g = K$$

We know that ENERGY will never change. As the rocket leaves the earth it's kinetic is large and its potential is small. As it ascends, there is a transfer of energy such that the difference between the kinetic and potential will always equal to **ZERO**.

Escape Speed

$$\frac{GmM}{r} = \frac{1}{2}mv^2$$

$$v_{escape} = \sqrt{\frac{2GM}{R}}$$

This expression is called the escape speed!

Due to the rotation of the earth, we can take advantage of the fact that we are rotating at a speed of 1500 km/h at the Cape!

NOTE: THIS IS ONLY FOR A SYSTEM WHERE YOU ARE TRYING TO GET THE OBJECT IN ORBIT!!!!!

Kepler's Laws

There are three laws that Johannes Kepler formulated when he was studying the heavens

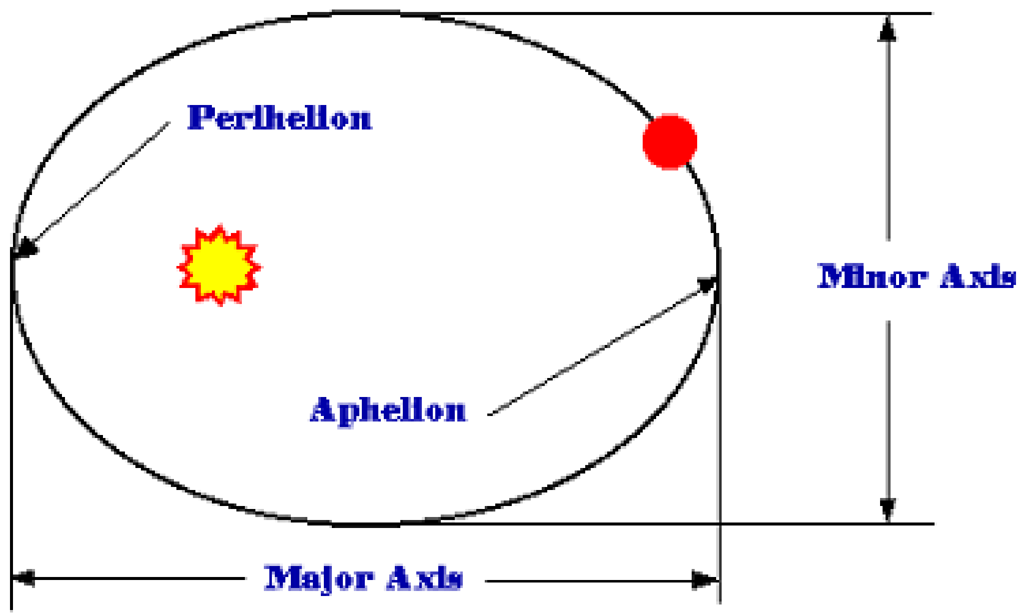
THE LAW OF ORBITS - "*All planets move in elliptical orbits, with the Sun at one focus.”*

THE LAW OF AREAS - "*A line that connects a planet to the sun sweeps out equal areas in the plane of the planet's orbit in equal times, that is, the rate dA/dt at which it sweeps out area A is constant.”*

THE LAW OF PERIODS - "*The square of the period of any planet is proportional to the cube of the semi major axis of its orbit."*

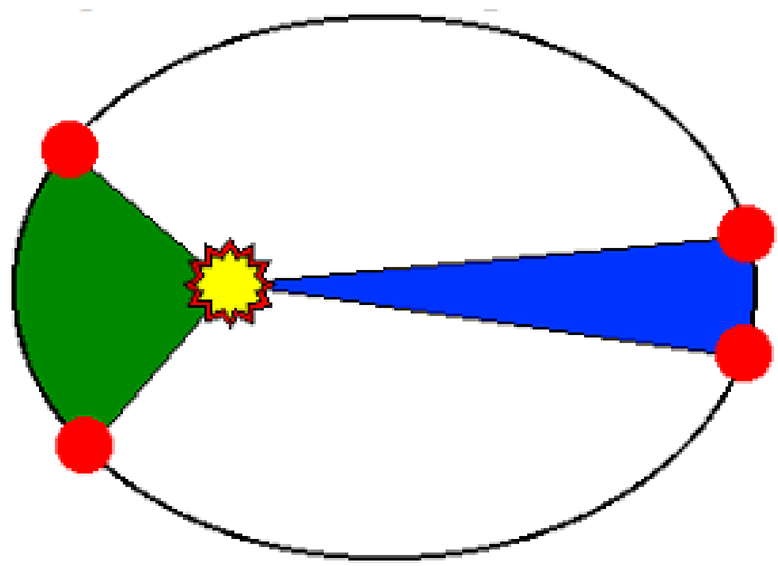
Kepler's 1st law – The Law of Orbits

"*All planets move in elliptical orbits, with the Sun at one focus.*"



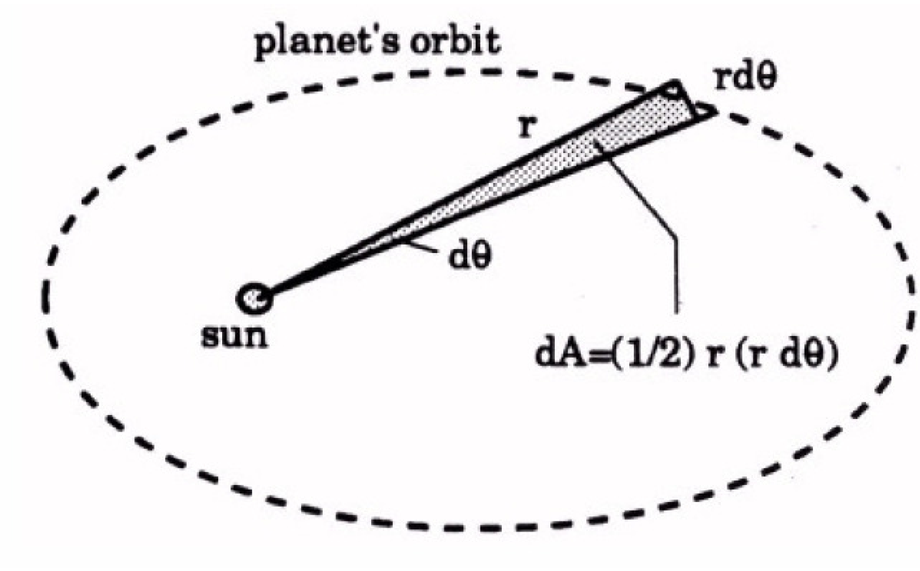
Kepler's 2^{nd} Law – The Law of Areas

"*A line that connects a planet to the sun sweeps out equal areas in the plane of the planet's orbit in equal times, that is, the rate dA/dt at which it sweeps out area A is constant.*"



Kepler's 2nd Law

How do we know that the rate at which the area is swept is constant?



$$Base = rd\theta, height = r$$

$$Area = \frac{1}{2}r(rd\theta)$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}, \frac{d\theta}{dt} = \omega$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega$$

$$L = pxr = mv\sin\theta r, v = r\omega$$

$$L = mr^2\omega$$

Angular momentum is conserved and thus constant! We see that both are proportional to the same two variables, thus Kepler's second law holds true to form.

Kepler's 3rd Law – The Law of Periods

"The square of the period of any planet is proportional to the cube of the semi major axis of its orbit."

$$F_g = G\frac{mM}{r^2}, F_c = \frac{mv^2}{r}$$

Gravitational forces are centripetal, thus we can set them equal to each other!

$$F_g = F_c$$

$$G\frac{mM}{r^2} = \frac{mv^2}{r}$$

Since we are moving in a circle we can substitute the appropriate velocity formula!

$$G\frac{M}{r} = v^2, v = \frac{2\pi r}{T}$$

The expression in the RED circle derived by setting the centripetal force equal to the gravitational force is called **ORBITAL SPEED**.

$$G\frac{M}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

Using algebra, you can see that everything in the parenthesis is CONSTANT. Thus the proportionality holds true!

Kinetic Energy in Orbit

$$G\frac{M}{r} = v^2, K = \frac{1}{2}mv^2$$

$$K = \frac{GmM}{2r}$$

Using our ORBITAL SPEED derived from K.T.L and the formula for kinetic energy we can define the kinetic energy of an object in a bit more detail when it is in orbit around a body.

The question is WHY? Why do we need a new equation for kinetic energy? Well, the answer is that greatly simplifies the math. If we use regular kinetic energy along with potential, we will need both the orbital velocity AND the orbital radius. In this case, we need only the orbital radius.

Total Energy of an orbiting body

$$E_{total} = K + U$$

$$U = -\frac{GmM}{r}$$

$$E_{Total} = G\frac{mM}{2r} + (-G\frac{mM}{2}) = -G\frac{mM}{2r}$$

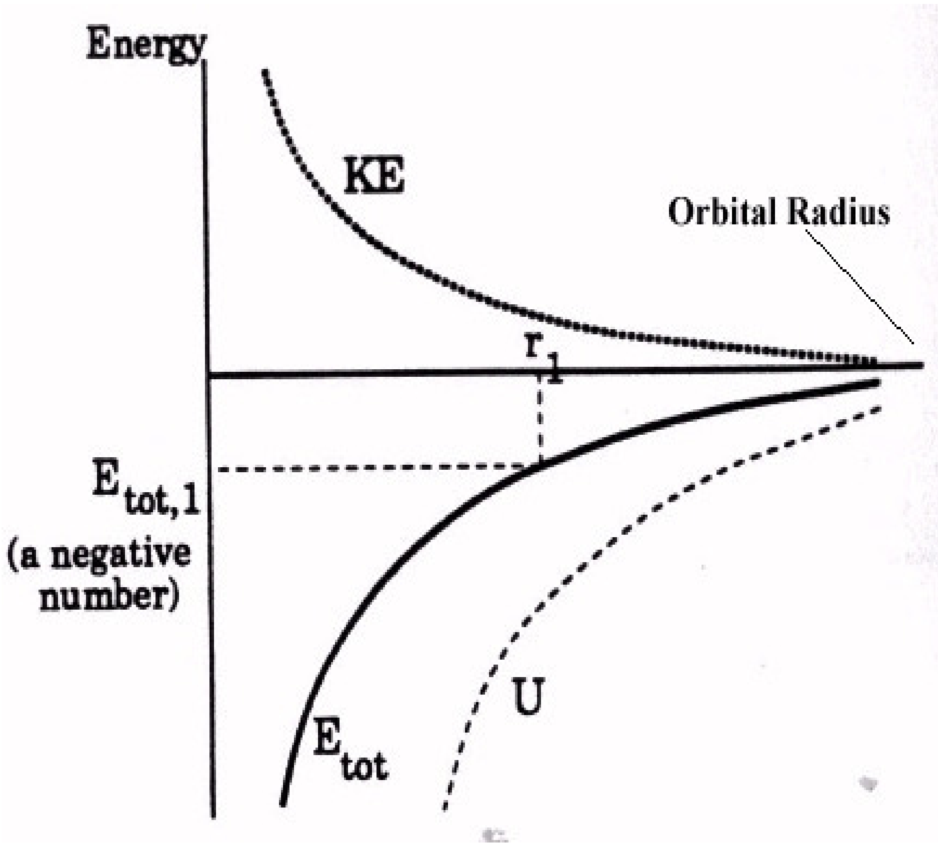
Notice the lack of velocities in this expression as mentioned in the last slide.

So by inspection we see that the kinetic energy function is always positive, the potential is negative and the total energy function is negative.

In fact the total energy equation is the negative inverse of the kinetic.

The negative is symbolic because it means that the mass “m” is BOUND to the mass of “M” and can never escape from it. It is called a BINDING ENERGY.

Energy from a graphical perspective



As the radius of motion gets larger. The orbiting body's kinetic energy must decrease (slows down) and its potential energy must increase (become less negative).

By saying become less negative means that we have defined our ZERO position for our potential energy at INFINITY.

Please make your selection...

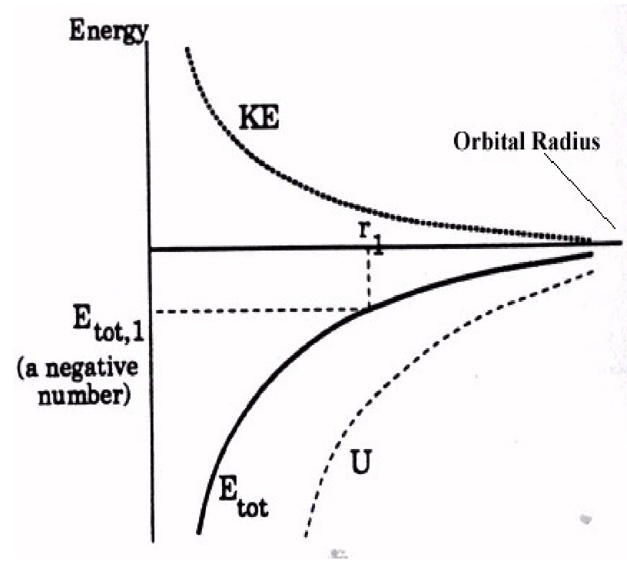
How do you move into a higher velocity orbit?

Question: If we have an orbiting Earth satellite and we want to put it in a higher velocity orbit, how can we use the satellite's thrusters to make the adjustment?

a) Fire Backwards

b) Fire Forwards

Backwards = speed up
Forwards = slow down



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30										

Fastest Responders (in seconds)

0	Participant 1
0	Participant 2
0	Participant 3
0	Participant 4
0	Participant 5

How do you move into a higher velocity orbit?

1) If you fire backwards thinking you will speed up the satellite you put it into a larger orbital radius which ultimately SLOWS DOWN the satellite as the KE decreases.
2) By thrusting backwards you are ADDING energy to the system moving the total energy closer to ZERO, this results in a larger radius which also causes the KE to decrease.
3) Fire forwards gently so that you do NEGATIVE WORK. This will cause the satellite to fall into a smaller orbit increasing the KE and increasing the speed. It also makes the potential energy increase negatively because you are moving farther from infinity. As the potential increase the KE again decreases.